Journal of Algebra 322 (2009) 2861–2885

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Buchsbaumness of certain generalization of the associated graded modules in the equi-$\mathbb{I}$-invariant case

Kikumichi Yamagishi

Faculty of Econoinformatics, Himeji Dokkyo University, Kamiono 7-2-1, Himeji, Hyogo 670-8524, Japan

ARTICLE INFO

Article history:
Received 21 February 2009
Available online 18 August 2009
Communicated by Steven Dale Cutkosky

Keywords:
Buchsbaum module
Associated graded module
$\mathbb{I}$-invariant

ABSTRACT

The present article introduces a generalization of the associated graded modules, writing $G_\alpha(\mathfrak{a}, E)$. Suppose that the equi-$\mathbb{I}$-invariant case occurs. Then it is shown that our graded module $G_\alpha(\mathfrak{a}, E)$ is Buchsbaum over $R(\mathfrak{a})$ for any $\alpha > 0$, if the given A-module E is so over A. Moreover, it is also shown that E is Buchsbaum over A iff $G_\alpha(\mathfrak{m}, E)$ is so over $R(\mathfrak{m})$ for all $\alpha > 0$, where $\mathfrak{m}$ is the maximal ideal of A.

1. Introduction

The aim of this paper is to study the Buchsbaumness of certain new graded modules, which are generalizations of the associated graded modules. Throughout this paper let $(A, \mathfrak{m})$ be a Noetherian local ring and E a finitely generated A-module of positive dimension s. We say that E is a *Buchsbaum* A-module, see [S-V], if the difference $l_A(E/\mathfrak{q}E) - e_\mathfrak{q}(E)$ is an invariant of E, so-called "the Buchsbaum invariant", not depending on the choice of parameter ideal $\mathfrak{q}$ of E, where $l_A(*)$ and $e_\mathfrak{q}(*)$ denote the length and the multiplicity with respect to $\mathfrak{q}$ (of an A-module), respectively. The local ring is called a Buchsbaum ring if it is a Buchsbaum module over itself.

We denote by $G(\mathfrak{a}, E)$ the associated graded module of E with respect to a proper ideal $\mathfrak{a}$ of A such that $l_A(E/\mathfrak{a}E) < \infty$; namely,

$$G(\mathfrak{a}, E) := \bigoplus_{n \geqslant 0} \mathfrak{a}^n E/\mathfrak{a}^{n+1} E.$$

Then $G(\mathfrak{a}, E)$ is a finitely generated graded $R(\mathfrak{a})$-module of dimension s, where $R(\mathfrak{a}) := \bigoplus_{n \geqslant 0} \mathfrak{a}^n$ denotes the Rees algebra of $\mathfrak{a}$. Now, we shall introduce a new graded modules as a generalization of

E-mail address: yamagisi@himeji-du.ac.jp.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.08.002

the associated graded module of E with respect to $\mathfrak{a}$. For any integer $\alpha > 0$, we define the following new graded $R(\mathfrak{a})$-module, writing $G_\alpha(\mathfrak{a}, E)$, as follows:

$$G_\alpha(\mathfrak{a}, E) := \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}^n E/\mathfrak{a}^{n+\alpha} E,$$

here we put $\mathfrak{a}^n = A$ for each $n \leqslant 0$; cf. [H-R-S, §2], see also [Y2]. It is obvious that $G_\alpha(\mathfrak{a}, E)$ is a finitely generated graded $R(\mathfrak{a})$-module of dimension s whose graduation is given by $[G_\alpha(\mathfrak{a}, E)]_n = \mathfrak{a}^n E/\mathfrak{a}^{n+\alpha} E$, in particular $[G_\alpha(\mathfrak{a}, E)]_n = (0)$ for $n \leqslant -\alpha$. Also $G_1(\mathfrak{a}, E) = G(\mathfrak{a}, E)$ clearly. We usually say that "$G_\alpha(\mathfrak{a}, E)$ is a Buchsbaum $R(\mathfrak{a})$-module", if $G_\alpha(\mathfrak{a}, E)_N$ is a Buchsbaum $R(\mathfrak{a})_N$-module, where $N := \mathfrak{m}R(\mathfrak{a}) + R(\mathfrak{a})_+$ denotes the unique homogeneous maximal ideal of $R(\mathfrak{a})$, as the same as [S-V, p. 200].

Let $\mathrm{H}^i_\mathfrak{m}(*)$ stand for the i-th local cohomology functor with respect to $\mathfrak{m}$. We denote by $h^i(E)$ the length of the i-th local cohomology module $\mathrm{H}^i_\mathfrak{m}(E)$ of E, i.e., $h^i(E) := l_A(\mathrm{H}^i_\mathfrak{m}(E))$. We define an invariant of E, writing $\mathbb{I}(E)$, as follows:

$$\mathbb{I}(E) := \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot h^i(E),$$

here recall $s := \dim_A E$. Then $0 \leqslant \mathbb{I}(E) \leqslant \infty$ clearly, and E is Cohen–Macaulay if and only if $\mathbb{I}(E) = 0$ holds. Since $\mathrm{H}^i_{NR(\mathfrak{a})_N}(G_\alpha(\mathfrak{a}, E)_N) \cong \mathrm{H}^i_N(G_\alpha(\mathfrak{a}, E))$, we simply write by $\mathbb{I}(G_\alpha(\mathfrak{a}, E))$ instead of $\mathbb{I}(G_\alpha(\mathfrak{a}, E)_N)$, if no confusions can be expected.

It is well known that $\mathbb{I}(G(\mathfrak{a}, E)) \geqslant \mathbb{I}(E)$ holds in general. In the case where the equality $\mathbb{I}(G(\mathfrak{a}, E)) = \mathbb{I}(E)$ holds (we call it here "the equi-$\mathbb{I}$-invariant case"), the author [Y2] shown that the associated graded module $G(\mathfrak{a}, E)$ is Buchsbaum $R(\mathfrak{a})$-module, if E itself is so over A. Inspired this argument, we shall try to show whether our new graded module $G_\alpha(\mathfrak{a}, E)$ defined above is Buchsbaum for any α. Namely we have the following.

Theorem (1.1). *Let E be a finitely generated A-module of dimension $s > 0$ and $\mathfrak{a}$ a proper ideal such that $l_A(E/\mathfrak{a}E) < \infty$. Suppose that E is a Buchsbaum A-module and the equality $\mathbb{I}(G(\mathfrak{a}, E)) = \mathbb{I}(E)$ holds. Then, for any $\alpha > 0$, the graded module $G_\alpha(\mathfrak{a}, E)$ is Buchsbaum over $R(\mathfrak{a})$ and the equality $\mathbb{I}(G_\alpha(\mathfrak{a}, E)) = \alpha \cdot \mathbb{I}(E)$ holds.*

There were already given many results about the Buchsbaumness of $G(\mathfrak{a}, E)$, under the hypothesis that E is Buchsbaum over A; cf. [G1,G2,Na], etc. On the other hand, the Buchsbaumness of the associated graded module $G(\mathfrak{a}, E)$ does not necessarily imply the same one of E itself in general. Concerning this topics, only few results were given up to now; see [G3], and also [G-Ni, Theorem 1.3]. However, our Theorem (1.1) inspire us to the following question.

Problem. Suppose that E has finite local cohomology and the equality $\mathbb{I}(G(\mathfrak{a}, E)) = \mathbb{I}(E)$ holds. If all the graded modules $G_\alpha(\mathfrak{a}, E)$ for $\alpha > 0$ are Buchsbaum (resp. quasi-Buchsbaum) over $R(\mathfrak{a})$, then does the A-module E itself have the same property?

Recall that E is said to have *finite local cohomology* if all the local cohomology modules $\mathrm{H}^i_\mathfrak{m}(E)$ of E for $i \neq s$ are of finite length, i.e., $l_A(\mathrm{H}^i_\mathfrak{m}(E)) < \infty$ for all $i \neq s$, cf. [G-Y]. Moreover, we say that E is a *quasi-Buchsbaum A-module*, if all of the local cohomology modules $\mathrm{H}^i_\mathfrak{m}(E)$ for $i \neq s$ of E are annihilated by the maximal ideal $\mathfrak{m}$ of A, i.e., $\mathfrak{m} \cdot \mathrm{H}^i_\mathfrak{m}(E) = (0)$ for all $i \neq s$. Note that Buchsbaum modules are always quasi-Buchsbaum modules and the Buchsbaum invariant of E coincides with $\mathbb{I}(E)$ defined as in the above.

We shall completely succeed in getting an affirmative answer to our problem about the quasi-Buchsbaumness. Namely we can state the following.

Theorem (1.2). *Let E be a finitely generated A-module of dimension $s > 0$ and $\mathfrak{a}$ a proper ideal of A such that $l_A(E/\mathfrak{a}E) < \infty$. Suppose that E has finite local cohomology and the equality $\mathbb{I}(G(\mathfrak{a}, E)) = \mathbb{I}(E)$ holds. Then the following two statements are equivalent.*

(1) *E is a quasi-Buchsbaum A-module.*
(2) *$G_\alpha(\mathfrak{a}, E)$ is a quasi-Buchsbaum $R(\mathfrak{a})$-module for all $\alpha > 0$.*

When this is the case, the equality $\mathbb{I}(G_\alpha(\mathfrak{a}, E)) = \alpha \cdot \mathbb{I}(E)$ also holds for each $\alpha > 0$.

However, for the Buchsbaumness, we can state only a partial answer to our problem before. Namely, in the case where a proper ideal $\mathfrak{a}$ coincides with the maximal ideal $\mathfrak{m}$ of A, we have the following.

Theorem (1.3). *Let E be a finitely generated A-module of positive dimension. Suppose that E has finite local cohomology and the equality $\mathbb{I}(G(\mathfrak{m}, E)) = \mathbb{I}(E)$ holds. Then the following two statements are equivalent.*

(1) *E is a Buchsbaum A-module.*
(2) *$G_\alpha(\mathfrak{m}, E)$ is a Buchsbaum $R(\mathfrak{m})$-module for all $\alpha > 0$.*

When this is the case the equality $\mathbb{I}(G_\alpha(\mathfrak{m}, E)) = \alpha \cdot \mathbb{I}(E)$ holds too for each $\alpha > 0$.

After preparing several basic facts on new graded module $G_\alpha(\mathfrak{a}, E)$, we shall prove our results. Firstly, we shall show Theorem (1.2) in Section 3. Secondly, Theorems (1.1) and (1.3) shall be proven in Section 4. Concerning Theorem (1.3), there was already given an interesting example, namely, a one-dimensional Noetherian local ring $(A, \mathfrak{m})$ such that the associated graded ring $G(\mathfrak{m}, A)$ is a Buchsbaum ring with the equality $\mathbb{I}(G(\mathfrak{m}, A)) = \mathbb{I}(A)$, but A itself is not so [G3]. Finally, in Section 5, we shall discuss the details of such example and we shall give some remarks.

2. Preliminaries

In this section, we shall discuss the behavior of unconditioned strong d-sequences on filtrations. Let A be a commutative ring, and E an A-module. A sequence $a_1, a_2, \dots, a_s$ ($s > 0$) of elements in A is said to be a *d-sequence* on E, see [H], if the equality

$$\mathfrak{q}_{i-1}E : a_i a_j = \mathfrak{q}_{i-1}E : a_j$$

holds for $1 \leqslant i \leqslant j \leqslant s$, here put $\mathfrak{q}_{i-1} = (a_1, a_2, \dots, a_{i-1})$ and $\mathfrak{q}_0 = (0)$, and moreover it is said to be an *unconditioned* d-sequence on E if it is still a d-sequence on E in any order.

Definition (2.1). (See [G-Y].) We will say that $a_1, a_2, \dots, a_s$ form an *unconditioned strong d-sequence* (abbrev. a u.s.d-sequence) on E if $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form a d-sequence on E in any order for every integer $n_1, n_2, \dots, n_s > 0$.

We need one more useful notation to describe our remarks on u.s.d-sequences. For a system of elements in A, say $a_1, a_2, \dots, a_s$, we define an A-submodule $\Sigma(a_1, \dots, a_s; E)$ of E as follows:

$$\Sigma(a_1, \dots, a_s; E) := \sum_{i=1}^{s} \big[(a_1, \dots, \widehat{a_i}, \dots, a_s)E : a_i\big] + (a_1, \dots, a_s)E,$$

where the hat $\widehat{\ }$ on a_i means to omit this element a_i from the system $a_1, a_2, \dots, a_s$. Moreover, for any two integers i, j, we denote by $[i, j]$ the set of integers n such that $i \leqslant n \leqslant j$. Of course, $[i, j] = \emptyset$ if $i > j$.

Here we shall collect several basic facts on unconditioned strong d-sequences, cf. [G-Y, §2]. Let $a_1, a_2, \dots, a_s$ be an unconditioned strong d-sequence on E. Then, for any positive integers m_i, n_i $(1 \leqslant i \leqslant s)$, the equality

$$\left(a_1^{m_1+n_1}, \dots, a_s^{m_s+n_s}\right)E : \prod_{i=1}^{s} a_i^{m_i} = \Sigma\left(a_1^{n_1}, \dots, a_s^{n_s}; E\right) \tag{$\sharp$2.1}$$

holds. Moreover, for any non-negative (not necessarily positive) integers l_i $(1 \leqslant i \leqslant s)$, the equality

$$\Sigma\left(a_1^{l_1+n_1}, \dots, a_s^{l_s+n_s}; E\right) : \prod_{i=1}^{s} a_i^{l_i} = \Sigma\left(a_1^{n_1}, \dots, a_s^{n_s}; E\right) \tag{$\sharp$2.2}$$

holds too. Finally, setting $\mathfrak{q} := (a_1, a_2, \dots, a_s)$, we also know that the equality

$$\left(a_i^{n_i} \mid i \in I\right)E \cap \mathfrak{q}^n E = \sum_{i \in I} a_i^{n_i} \mathfrak{q}^{n-n_i} E \tag{$\sharp$2.3}$$

holds for all $I \subseteq [1, s]$, $n_i > 0$ and $n \in \mathbb{Z}$. In fact, the assertion ($\sharp$2.1) (resp. ($\sharp$2.3)) is given in the same way as in the proof of [Y1, Lemma 1.2] (resp. [G2, Corollary (1.2)]), and see also [G-Y, Theorems (2.3) and (2.6)] for more explicit statements.

A family $\{F_n\}_{n \in \mathbb{Z}}$ of A-submodules of E is called a $\mathfrak{q}$-*filtration* of E if $F_n \supseteq F_{n+1} \supseteq \mathfrak{q}F_n$ for all n and $F_0 = E$. Then $\mathfrak{q}^m F_n \subseteq F_{m+n}$ for all $m, n \in \mathbb{Z}$ clearly. Now we begin with the following, which is a generalization of Lemma (2.5) in [G-Y].

Proposition (2.2). *Let $\{F_n\}_{n \in \mathbb{Z}}$ be a $\mathfrak{q}$-filtration of E, where $\mathfrak{q} := (a_1, a_2, \dots, a_s)$. Suppose that there is an integer $m > 0$ such that $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned d-sequence on E for all $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$. Then the following two statements are equivalent.*

(1) *The equality*

$$\left(a_i^{n_i} \mid i \in I\right)E \cap F_n = \sum_{i \in I} a_i^{n_i} F_{n-n_i}$$

holds for all $I \subseteq [1, s]$, $n \in \mathbb{Z}$ and $n_i \geqslant m$.

(2) *The equality*

$$\left(a_1^{2m}, a_2^{2m}, \dots, a_s^{2m}\right)E \cap F_n = \left(a_1^{2m}, a_2^{2m}, \dots, a_s^{2m}\right)F_{n-2m}$$

holds for all $n \in \mathbb{Z}$.

First of all, we begin with the following claim:

Lemma (2.3). *Suppose that the statement* (2) *of Proposition* (2.2) *holds. Then one has the following.*

(i) *If $s \geqslant 2$, then the equality*

$$\left(a_1^{2m}, \dots, a_{s-1}^{2m}\right)E \cap F_n = \left(a_1^{2m}, \dots, a_{s-1}^{2m}\right)F_{n-2m}$$

holds for all $n \in \mathbb{Z}$.

(ii) *The equality*

$$\big(a_1^{2m},\dots,a_{s-1}^{2m},a_s^k\big)E\cap F_n=\big(a_1^{2m},\dots,a_{s-1}^{2m}\big)F_{n-2m}+a_s^kF_{n-k}$$

holds for all $m\leqslant k<2m$ and $n\in\mathbb{Z}$. (Notice that, in the case $s=1$, we set $(a_1^{2m},\dots,a_{s-1}^{2m})=(0)$.)

(iii) $a_1^kE\cap F_n=a_1^kF_{n-k}$ *holds for all $k\geqslant m$ and $n\in\mathbb{Z}$.*

Proof. (i) We may assume that $n>2m$ and that our assertion is true for any integer less than n. Let $x\in(a_1^{2m},\dots,a_{s-1}^{2m})E\cap F_n$. Then $x\in(a_1^{2m},\dots,a_s^{2m})E\cap F_n$ clearly. By our assumption (2) we have

$$x=y+a_s^{2m}z$$

for suitable elements $y\in(a_1^{2m},\dots,a_{s-1}^{2m})F_{n-2m}$ and $z\in F_{n-2m}$. Since $a_1^m,a_2^m,\dots,a_s^m$ is a u.s.d-sequence on E, we have $a_s^mz\in(a_1^{2m},\dots,a_{s-1}^{2m})E\cap F_{n-m}$. Applying the hypothesis of induction on n we get $a_s^mz\in(a_1^{2m},\dots,a_{s-1}^{2m})F_{n-3m}$. By the expression of x above, we finally conclude $x\in(a_1^{2m},\dots,a_{s-1}^{2m})F_{n-2m}$.

(ii) Let $x\in(a_1^{2m},\dots,a_{s-1}^{2m},a_s^k)E\cap F_n$. Then $a_s^{2m-k}x\in(a_1^{2m},\dots,a_s^{2m})E\cap F_{n+2m-k}$. By our assumption (2) we see

$$a_s^{2m-k}x=y+a_s^{2m}z,$$

where $y\in(a_1^{2m},\dots,a_{s-1}^{2m})F_{n-k}$ and $z\in F_{n-k}$. Applying the assertion (i), we see

$$\begin{aligned}x-a_s^kz&\in\big[\big(a_1^{2m},\dots,a_{s-1}^{2m}\big)E:a_s^{2m-k}\big]\cap\big(a_1^{2m},\dots,a_{s-1}^{2m},a_s^k\big)E\cap F_n\\&\subseteq\big[\big(a_1^{2m},\dots,a_{s-1}^{2m}\big)E:a_s^{k}\big]\cap\big(a_1^{2m},\dots,a_{s-1}^{2m},a_s^k\big)E\cap F_n\\&=\big(a_1^{2m},\dots,a_{s-1}^{2m}\big)E\cap F_n=\big(a_1^{2m},\dots,a_{s-1}^{2m}\big)F_{n-2m},\end{aligned}$$

because $1\leqslant 2m-k\leqslant m\leqslant k$ and $a_1^{2m},\dots,a_{s-1}^{2m},a_s^k$ form a d-sequence on E. Thus we get $x\in(a_1^{2m},\dots,a_{s-1}^{2m})F_{n-2}+a_s^kF_{n-k}$.

(iii) The statement (2) of Proposition (2.2) implies that $a_1^{2m}E\cap F_n=a_1^{2m}F_{n-2m}$ for all $n\in\mathbb{Z}$ by the assertion (i). On the other hand, since $a_1^{n_1}$ is a d-sequence on E for $n_1\geqslant m$ consisting of a single element, we can use the assertion (ii) setting $s=1$. Namely we have that

$$a_1^kE\cap F_n=a_1^kF_{n-k}$$

for $m\leqslant k<2m$ too. So we may assume that $k>2m$ and our assertion is true for $k-1$. Write $a:=a_1$. Let $x\in a^kE\cap F_n$. Then x has an expression such that $x=a^ky$ with $y\in E$. By the hypothesis of induction on k, we see $a^kE\cap F_n\subseteq a^{k-1}E\cap F_n=a^{k-1}F_{n-k+1}$. Hence x has another expression as follows $x=a^{k-1}z$ with $z\in F_{n-k+1}$. Since $k>2m$ we know $k-m-1\geqslant m$. Combining these observations we have

$$a^{k-m}y-a^{k-m-1}z\in\big[0:a^m\big]\cap a^mE=(0)$$

by the d-sequence property of a^m. This means $a^{k-m}y=a^{k-m-1}z$, and hence $a^{k-m}y\in F_{n-m}$. Thus we know $a^{k-m}y\in a^{k-m}E\cap F_{n-m}=a^{k-m}F_{n-k}$ by the hypothesis of induction on k again. Therefore we finally get $x=a^ma^{k-m}y\in a^kF_{n-k}$ and this completes the proof of Lemma (2.3). $\square$

Now we are ready to prove our proposition.

Proof of Proposition (2.2). It is enough to prove the implication (2) $\Longrightarrow$ (1). By (i) we may assume that $I=[1,s]$. We shall apply the similar argument as in [G2, proof of (1.2)]. Namely assume that this implication fails to hold and choose $t := n + \sum_{i=1}^{s} n_i$ as small as possible among such counterexamples, say

$$\left(a_1^{n_1}, \dots, a_s^{n_s}\right)E \cap F_n \not\subseteq \sum_{i=1}^{s} a_i^{n_i} F_{n-n_i}.$$

By (iii) of Lemma (2.3), we have $s \geqslant 2$. Moreover we know that $n_i \geqslant 2m$ for all i. In fact, if $n_i < 2m$ for some i, say $n_s < 2m$. Then applying the minimality of t to $E/a_s^{n_s}E$, we see

$$\begin{aligned}\left(a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}, a_s^{n_s}\right)E \cap F_n &= \sum_{i=1}^{s-1} a_i^{n_i} F_{n-n_i} + \left(a_s^{n_s} E \cap F_n\right) \\ &= \sum_{i=1}^{s-1} a_i^{n_i} F_{n-n_i} + a_s^{n_s} F_{n-n_s},\end{aligned}$$

by (ii) and (iii) of Lemma (2.3), so this contradicts our choice of t. Moreover, by our assumption (2), we know that $n_i > 2m$ for some i, say $n_s > 2m$. Let $x \in (a_1^{n_1}, \dots, a_s^{n_s})E \cap F_n$ such that $x \notin \sum_{i=1}^{s} a_i^{n_i} F_{n-n_i}$. Then this element x can be written as follows:

$$x = y + a_s^{n_s} z$$

with $y \in (a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}})E$ and $z \in E$. Notice that $x \in (a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}, a_s^{n_s-m})E \cap F_n$. By the minimality of t, we find another expression of x such that

$$x = u + a_s^{n_s-m} v$$

with $u \in \sum_{i=1}^{s-1} a_i^{n_i} F_{n-n_i}$ and $v \in F_{n-n_s+m}$. Since $n_s > 2m$ we know $n_s - m > m$. Comparing both expressions of x, we have

$$\begin{aligned} v - a_s^m z &\in \left(a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}\right)E : a_s^{n_s-m} \\ &\subseteq \left(a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}\right)E : a_s^{mk} \\ &= \left(a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}\right)E : a_s^{m},\end{aligned}$$

where $mk \geqslant n_s - m$ for some $k > 0$. Hence

$$a_s^m v \in \left(a_1^{n_1}, \dots, a_{s-1}^{n_{s-1}}, a_s^{2m}\right)E \cap F_{n-n_s+2m}.$$

Since $n_s > 2m$ again, we see $(n_1 + \cdots + n_{s-1} + 2m) + (n - n_s + 2m) = t - 2(n_s - 2m) < t$, and hence by the minimality of t again we have $a_s^m v \in \sum_{i=1}^{s-1} a_i^{n_i} F_{n-n_i-n_s+2m} + a_s^{2m} F_{n-n_s}$. Therefore we obtain

$$x = u + a_s^{n_s-2m} a_s^m v \in \sum_{i=1}^{s} a_i^{n_i} F_{n-n_i},$$

and this is a contradiction to our choice of t. This finishes the proof of Proposition (2.2). $\square$

Corollary (2.4). *Let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a} \supseteq (a_1, a_2, \dots, a_s)$. Suppose $m > 0$ is an integer such that $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form an unconditioned d-sequence on E for all $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$. Then the following two statements are equivalent.*

(1) *The equality*

$$\left(a_i^{n_i} \mid i \in I\right)E \cap \mathfrak{a}^n E = \sum_{i \in I} a_i^{n_i} \mathfrak{a}^{n-n_i} E$$

holds for all $I \subseteq [1, s]$, $n \in \mathbb{Z}$ *and* $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$.

(2) *The equality*

$$\left(a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m}\right)E \cap \mathfrak{a}^n E = \left(a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m}\right)\mathfrak{a}^{n-2m} E$$

holds for all $n \in \mathbb{Z}$.

When this is the case one also has a graded R*-isomorphism*

$$G_\alpha(\mathfrak{a}, E)/(a_i t)^{n_i} \cdot G_\alpha(\mathfrak{a}, E) \cong G_\alpha\left(\mathfrak{a}, E/a_i^{n_i} E\right)$$

for any $n_i \geqslant m$ *and* $1 \leqslant i \leqslant s$.

3. The quasi-Buchsbaumness of $G_\alpha(\mathfrak{a}, E)$'s

Throughout this section, let A denote a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let E be a finitely generated A-module of positive dimension. Let us write $s := \dim_A E$. For simplicity, we always assume that the residue field $A/\mathfrak{m}$ is infinite.

For a system of parameters $a_1, a_2, \ldots, a_s$ for E, we define that

$$\mathbb{I}(a_1, a_2, \ldots, a_s; E) := l_A(E/\mathfrak{q}E) - e_0(\mathfrak{q}; E),$$

where $e_0(\mathfrak{q}; E)$ denotes the multiplicity of E with respect to $\mathfrak{q} = (a_1, a_2, \ldots, a_s)$.

Let us keep the notation $h^i(E)$ as same as before and let us recall the definition of $\mathbb{I}(E)$ as follows:

$$\mathbb{I}(E) := \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot h^i(E).$$

We say that E has *finite local cohomology* if $h^i(E) < \infty$ for all $i \neq s$, cf. [G-Y], and this is equivalent to saying $\mathbb{I}(E) < \infty$. Recall that $\operatorname{Supp}_A E$ is said to be *catenary* if, for each pair of prime ideals $\mathfrak{p}$, $\mathfrak{p}'$ such that $\mathfrak{p} \supset \mathfrak{p}'$, any maximal prime chain between $\mathfrak{p}$ and $\mathfrak{p}'$ has the same length $(= \operatorname{ht} \mathfrak{p}/\mathfrak{p}')$, and that $\operatorname{Supp}_A E$ is said to be *equidimensional* if $\dim A/\mathfrak{p} = \dim_A E$ for every minimal prime ideal $\mathfrak{p} \in \operatorname{Supp}_A E$.

Remark (3.1). (See [S-T-C].) (1) If E has finite local cohomology, then $E_\mathfrak{p}$ is Cohen–Macaulay $A_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Supp}_A E \setminus \{\mathfrak{m}\}$ and $\operatorname{Supp}_A E$ is catenary and equidimensional. Moreover, in the case where A is a homomorphic image of a Cohen–Macaulay local ring, E has finite local cohomology if and only if $E_\mathfrak{p}$ is Cohen–Macaulay $A_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Supp}_A E \setminus \{\mathfrak{m}\}$ and $\operatorname{Supp}_A E$ is equidimensional.

(2) E has finite local cohomology if and only if there is a system of parameters $a_1, a_2, \ldots, a_s$ for E such that

$$\sup_{n_1, n_2, \ldots, n_s > 0} \mathbb{I}\left(a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s}; E\right) < \infty.$$

When this is the case $\mathbb{I}(E)$ is the upper bound of all the $\mathbb{I}(a_1, a_2, \dots, a_s; E)$'s, where $a_1, a_2, \dots, a_s$ is any system of parameters for E, and moreover there is an integer $t > 0$ such that

$$\mathbb{I}(a_1, a_2, \dots, a_s; E) = \mathbb{I}(E)$$

for every system of parameters $a_1, a_2, \dots, a_s$ for E contained in $\mathfrak{m}^t$.

(3) For any system of parameters $a_1, a_2, \dots, a_s$ for E the following three conditions are equivalent:

(i) $a_1, a_2, \dots, a_s$ form an unconditioned strong d-sequence on E;
(ii) the equality $\mathbb{I}(a_1, a_2, \dots, a_s; E) = \mathbb{I}(E)$ holds;
(iii) the equality $\mathbb{I}(a_1, a_2, \dots, a_s; E) = \mathbb{I}(a_1^2, a_2^2, \dots, a_s^2; E)$ holds.

(Cf. [S, Theorem A], [T, Theorem 2.1]; see also [G-Y, Corollary (6.18)].)

Now, let $\mathfrak{a}$ be a proper ideal of A such that $l_A(E/\mathfrak{a}E) < \infty$. Notice that this is equivalent to saying that the ideal $\mathfrak{a} + \mathrm{Ann}_A(E)$ is $\mathfrak{m}$-primary in A. Untill the end of this section, let us keep the following notations:

$R := R(\mathfrak{a}) = \bigoplus_{n \geqslant 0} \mathfrak{a}^n$, the Rees algebra of $\mathfrak{a}$;
$N := \mathfrak{m}R + R_+$, the unique homogeneous maximal ideal of R;
$G(E) := G(\mathfrak{a}, E)$, the associated graded module of E with respect to $\mathfrak{a}$;
$G_\alpha(E) := G_\alpha(\mathfrak{a}, E)$, defined in Section 1.

Usually we regard the Rees algebra R as the A-subalgebra of the polynomial ring $A[t]$ over A with an indeterminate t such as $R = A[\{at \mid a \in \mathfrak{a}\}] \subset A[t]$.

Let us recall several basic facts from the theory of graded modules; see [G-W] for unexplained terminologies on the graded rings/modules. Let $W := \bigoplus_{n \in \mathbb{Z}} W_n$ be a graded module over R. We sometimes denote by $[W]_n$ the homogeneous component of degree n instead of W_n. Let r be an integer. We denote by $W(r)$ the shifted module of W of degree r, namely its underlying R-module coincides with W and its graduation is given by $[W(r)]_n = W_{n+r}$ for $n \in \mathbb{Z}$. Moreover, we denote by $W_{\geqslant r}$ the graded R-submodule of W given by $W_{\geqslant r} := \bigoplus_{n \geqslant r} W_n$. In particular we usually denote by W_+ instead of $W_{\geqslant 1}$. Since $R/N \cong A/\mathfrak{m}$, the unique homogeneous maximal ideal N of R is just a maximal ideal of R and for any graded R-module W the canonical R-homomorphism $W \longrightarrow W_N$ must be injective, where W_N denotes the localization of W at N.

Let $\mathfrak{q}$ be a parameter ideal of E such that $\mathfrak{q} \subseteq \mathfrak{a}$, and suppose that $\mathfrak{a}^{r+1}E = \mathfrak{q}\mathfrak{a}^r E$ for some integer $r \geqslant 0$. Write $\mathfrak{q} = (a_1, a_2, \dots, a_s)$. We set $f_i := a_i t$ in R_1 for each $1 \leqslant i \leqslant s$ and $Q := (f_1, f_2, \dots, f_s)R$. Since $G_\alpha(E)_{\geqslant r+1} \subseteq Q \cdot G_\alpha(E)$, the graded R-module $G_\alpha(E)/Q \cdot G_\alpha(E)$ is of finite length, and hence the ideal QR_N is a parameter ideal of $G_\alpha(E)_N$ over R_N and $\dim_R G_\alpha(E) = \dim_{R_N} G_\alpha(E)_N = s$.

Let $\mathrm{H}^i_N(*)$ stand for the i-th (graded) local cohomology functor with respect to N, cf. see [G-W] for more detail. We denote by $h^i(G_\alpha(E))$ the length of $\mathrm{H}^i_N(G_\alpha(E))$ over R, i.e., $h^i(G_\alpha(E)) := l_R(\mathrm{H}^i_N(G_\alpha(E)))$, and also define an invariant, say $\mathbb{I}(G_\alpha(E))$, in the same way as $\mathbb{I}(E)$ before, namely

$$\mathbb{I}\big(G_\alpha(E)\big) := \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot h^i\big(G_\alpha(E)\big),$$

here recall $\dim_R G_\alpha(E) = \dim_A E = s$. Notice that there is a canonical isomorphism $\mathrm{H}^i_N(G_\alpha(E)) \cong \mathrm{H}^i_{NR_N}(G_\alpha(E)_N)$ for each i, hence this new invariant coincides with the original $\mathbb{I}$-invariant $\mathbb{I}(G_\alpha(E)_N)$, namely $\mathbb{I}(G_\alpha(E)) = \mathbb{I}(G_\alpha(E)_N)$ holds. As described in Section 1, we simply say that $G_\alpha(E)$ has finite local cohomology over R if the localization $G_\alpha(E)_N$ at N has so over R_N.

Let $\alpha \geqslant 2$. Since $[G(E)(\alpha-1)]_n = \mathfrak{a}^{n+\alpha-1}E/\mathfrak{a}^{n+\alpha}E$, where $n \in \mathbb{Z}$, there is a short exact sequence of graded R-modules as follows:

$$0 \longrightarrow G(E)(\alpha-1) \longrightarrow G_\alpha(E) \longrightarrow G_{\alpha-1}(E) \longrightarrow 0. \tag{$\sharp$3.1}$$

From ($\sharp$3.1) we have

$$\mathrm{Ass}_R\, G(E) \subseteq \mathrm{Ass}_R\, G_\alpha(E) \subseteq \mathrm{Ass}_R\, G(E) \cup \mathrm{Ass}_R\, G_{\alpha-1}(E).$$

Using induction on α, we easily see that $\mathrm{Ass}_R\, G_\alpha(E) = \mathrm{Ass}_R\, G(E)$ and $\mathrm{Supp}_R\, G_\alpha(E) = \mathrm{Supp}_R\, G(E)$ for all $\alpha \geqslant 2$, because of $G_1(E) = G(E)$. On the other hand, applying the local cohomology functors to ($\sharp$3.1), we have the long exact sequence of local cohomology modules as follows:

$$\cdots \longrightarrow \mathrm{H}^i_N\big(G(E)\big)(\alpha-1) \longrightarrow \mathrm{H}^i_N\big(G_\alpha(E)\big) \longrightarrow \mathrm{H}^i_N\big(G_{\alpha-1}(E)\big) \longrightarrow \cdots.$$

Hence, we know that

$$h^i\big(G_\alpha(E)\big) \leqslant h^i\big(G(E)\big) + h^i\big(G_{\alpha-1}(E)\big)$$

and

$$h^i\big(G(E)\big) \leqslant h^i\big(G_\alpha(E)\big) + h^{i-1}\big(G_{\alpha-1}(E)\big)$$

for all i. By these observations, we get the following lemma.

Lemma (3.2). *Suppose that $\alpha \geqslant 2$. Then*:

(1) $\mathbb{I}(G_\alpha(E)) \leqslant \mathbb{I}(G(E)) + \mathbb{I}(G_{\alpha-1}(E)) \leqslant \alpha \cdot \mathbb{I}(G(E))$ *holds*;
(2) *if $G(E)$ has finite local cohomology, then $G_\alpha(E)$ has so*;
(3) *if $G_\alpha(E)$ and $G_{\alpha-1}(E)$ have finite local cohomology for some α, then $G(E)$ also has so*;
(4) $\mathrm{Ass}_R\, G_\alpha(E) = \mathrm{Ass}_R\, G(E)$ *and* $\mathrm{Supp}_R\, G_\alpha(E) = \mathrm{Supp}_R\, G(E)$ *hold.*

At the end of this section we shall show the converse of (2) of Lemma (3.2) as the above. Recall that $f_1, f_2, \ldots, f_s$ is a homogeneous system of parameters for $G_\alpha(E)_N$. We usually denote by $\mathbb{I}(f_1, f_2, \ldots, f_s; G_\alpha(E))$ instead of $\mathbb{I}(f_1, f_2, \ldots, f_s; G_\alpha(E)_N)$, if no confusion can be expected. With this notation we have the following.

Lemma (3.3). *Let $n_1, \ldots, n_s$ and α be any positive integers. Then*:

(1) $\mathbb{I}(f_1^{n_1}, f_2^{n_2}, \ldots, f_s^{n_s}; G_\alpha(E)) \geqslant \alpha \cdot \mathbb{I}(a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s}; E)$ *holds, and therefore one also has* $\mathbb{I}(G_\alpha(E)) \geqslant \alpha \cdot \mathbb{I}(E)$;
(2) *the equality* $\mathbb{I}(f_1^{n_1}, f_2^{n_2}, \ldots, f_s^{n_s}; G_\alpha(E)) = \alpha \cdot \mathbb{I}(a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s}; E)$ *holds if and only if*

$$\big(a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s}\big)E \cap \mathfrak{a}^n E = \sum_{i=1}^{s} a_i^{n_i} \mathfrak{a}^{n-n_i} E$$

holds for all $n \geqslant 0$.

Therefore, if the equality described as in the statement (2) *holds true for some α, then it does so for any α.*

Proof. These are shown by the similar way as in [T, Lemma 5.1] or [G-Y, Lemma (7.7)]. □

Proposition (3.4). *Suppose that E has finite local cohomology and there is an integer $m > 0$ such that $a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s}$ form an unconditioned d-sequence on E for all $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$. Then the following statements are equivalent.*

(1) *The equality $\mathbb{I}(G(E)) = \mathbb{I}(E)$ holds.*
(2) *The equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ holds for all $\alpha > 0$.*
(3) *The equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ holds for some $\alpha \geqslant 2$.*
(4) *The equality $(a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m})E \cap \mathfrak{a}^n E = (a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m})\mathfrak{a}^{n-2m}E$ holds for all $2m < n \leqslant s(2m-1)+r$.*

When this is the case, $f_1^{n_1}, f_2^{n_2}, \ldots, f_s^{n_s}$ form an unconditioned d-sequence on $G_\alpha(E)$ for all $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$ and $\alpha > 0$. Moreover, $E/a_i^{n_i}E$, for $n_i \geqslant m$ and $1 \leqslant i \leqslant s$, satisfies these equivalent conditions too (here note that the element a_i should be removed from the sequence $a_1, a_2, \ldots, a_s$), if $s \geqslant 2$.

Proof. Since $a_1^m, a_2^m, \ldots, a_s^m$ form a u.s.d-sequence on E, we have

$$\mathbb{I}\big(G_\alpha(E)\big) \geqslant \mathbb{I}\big(f_1^{ml_1}, \ldots, f_s^{ml_s}; G_\alpha(E)\big) \geqslant \alpha \cdot \mathbb{I}\big(a_1^{ml_1}, \ldots, a_s^{ml_s}; E\big) = \alpha \cdot \mathbb{I}(E)$$

for all $l_i > 0$ $(1 \leqslant i \leqslant s)$ and $\alpha > 0$; cf. Remark (3.1) and (1) of Lemma (3.3). Since for any $n > s(2m-1)+r$ we have

$$\mathfrak{a}^n = \mathfrak{q}^{n-r}\mathfrak{a}^r \subseteq \big(a_1^{2m}, \ldots, a_s^{2m}\big)\mathfrak{q}^{n-r-2m}\mathfrak{a}^r \subseteq \big(a_1^{2m}, \ldots, a_s^{2m}\big)\mathfrak{a}^{n-2m},$$

the assertion (4) is equivalent to saying that the equality

$$\big(a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m}\big)E \cap \mathfrak{a}^n E = \big(a_1^{2m}, a_2^{2m}, \ldots, a_s^{2m}\big)\mathfrak{a}^{n-2m}E$$

holds for all $n \in \mathbb{Z}$. Hence, by Corollary (2.4), (1) of Lemma (3.2) and (2) of Lemma (3.3), these observations imply the required equivalences.

Moreover, for any $n_i \geqslant m$ $(1 \leqslant i \leqslant s)$ and $\alpha > 0$, $f_1^{n_1}, f_2^{n_2}, \ldots, f_s^{n_s}$ form a u.s.d-sequence on $G_\alpha(E)_N$, clearly. Recall that, for any graded R-module W, the canonical R-homomorphism $W \longrightarrow W_N$ is injective. Thus $f_1^{n_1}, f_2^{n_2}, \ldots, f_s^{n_s}$ form a u.s.d-sequence on $G_\alpha(E)$ too.

Finally, to show the rest of our assertions, we put $n = n_i$. Then $\mathfrak{a}^{r+n}E = \mathfrak{q}^n\mathfrak{a}^r E = (a_1, \ldots, \widehat{a_i}, \ldots, a_s)\mathfrak{a}^{r+n-1}E + a_i^n\mathfrak{a}^r E$ clearly. Hence applying our arguments described as the before to $E/a_i^n E$, the sequence $a_1, \ldots, \widehat{a_i}, \ldots, a_s$ and $\mathfrak{a}$, we get that $E/a_i^n E$ for $n \geqslant m$ satisfies these equivalent conditions too. $\square$

Now put $U := \mathrm{H}_\mathfrak{m}^0(E)$ and $U^* := \operatorname{Ker}\varphi$, where $\varphi : G_\alpha(E) \longrightarrow G_\alpha(E/U)$ denotes the canonical epimorphism of graded R-modules via the projection $E \longrightarrow E/U$. Then there is a short exact sequence

$$0 \longrightarrow U^* \longrightarrow G_\alpha(E) \xrightarrow{\varphi} G_\alpha(E/U) \longrightarrow 0 \qquad (\sharp 3.2)$$

of graded R-modules. Since each homogeneous component of $G_\alpha(E/U)$ is given by

$$\big[G_\alpha(E/U)\big]_n = \mathfrak{a}^n E + U/\mathfrak{a}^{n+\alpha}E + U \cong \mathfrak{a}^n E/\big(U \cap \mathfrak{a}^n E\big) + \mathfrak{a}^{n+\alpha}E,$$

we have the following formula immediately:

$$[U^*]_n = \big(U \cap \mathfrak{a}^n E\big) + \mathfrak{a}^{n+\alpha}E/\mathfrak{a}^{n+\alpha}E \cong \big(U \cap \mathfrak{a}^n E\big)/\big(U \cap \mathfrak{a}^{n+\alpha}E\big), \qquad (\sharp 3.3)$$

for all $n \in \mathbb{Z}$, and hence $l_R(U^*) = \alpha \cdot l_A(U) = \alpha \cdot h^0(E)$. Now applying the local cohomology functors $\mathrm{H}^i_N(*)$ to the short exact sequence (♯3.2) we have the short exact sequence

$$0 \longrightarrow U^* \longrightarrow \mathrm{H}^0_N\bigl(G_\alpha(E)\bigr) \longrightarrow \mathrm{H}^0_N\bigl(G_\alpha(E/U)\bigr) \longrightarrow 0, \tag{♯3.4}$$

and the isomorphisms

$$\mathrm{H}^i_N\bigl(G_\alpha(E)\bigr) \cong \mathrm{H}^i_N\bigl(G_\alpha(E/U)\bigr) \quad \text{for all } i \geqslant 1, \tag{♯3.5}$$

because of $l_R(U^*) < \infty$. These observations (♯3.4) and (♯3.5) imply the following:

Lemma (3.5). *Let U and U^* be the same as above.*

(1) *The following equalities hold*:
$h^0(G_\alpha(E/U)) = h^0(G_\alpha(E)) - l_R(U^*) = h^0(G_\alpha(E)) - \alpha \cdot h^0(E)$, *and*
$h^i(G_\alpha(E/U)) = h^i(G_\alpha(E))$ *for all* $i \geqslant 1$.
(2) *The following three conditions are equivalent*:
(i) $h^0(G_\alpha(E/U)) = 0$;
(ii) $h^0(G_\alpha(E)) = \alpha \cdot h^0(E)$;
(iii) $\mathrm{H}^0_N(G_\alpha(E)) = U^*$.
(3) *Suppose that E has finite local cohomology. If the equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ holds, then the equality $\mathbb{I}(G_\alpha(E/U)) = \alpha \cdot \mathbb{I}(E/U)$ holds too.*

Proof. The assertions (1) and (2) are already shown above. Now assume that the equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ holds. By our assertion (1) we have

$$\mathbb{I}\bigl(G_\alpha(E/U)\bigr) = \mathbb{I}\bigl(G_\alpha(E)\bigr) - \alpha \cdot h^0(E) = \alpha\bigl(\mathbb{I}(E) - h^0(E)\bigr) = \alpha \cdot \mathbb{I}(E/U). \qquad \square$$

Proposition (3.6). *Let $\alpha > 0$. Suppose that E has finite local cohomology. Then the following two statements are equivalent.*

(1) *The equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ holds.*
(2) $h^i(G_\alpha(E)) = \alpha \cdot h^i(E)$ *for all* $0 \leqslant i < s$.

When this is the case, E/U, where $U := \mathrm{H}^0_{\mathfrak{m}}(E)$, also satisfies the equivalent condition. Moreover, the following sequence of graded R-modules

$$0 \longrightarrow \mathrm{H}^0_N\bigl(G_\alpha(E)\bigr) \longrightarrow G_\alpha(E) \longrightarrow G_\alpha(E/U) \longrightarrow 0$$

is exact and

$$\bigl[\mathrm{H}^0_N\bigl(G_\alpha(E)\bigr)\bigr]_n \cong \bigl(U \cap \mathfrak{a}^n E\bigr)/\bigl(U \cap \mathfrak{a}^{n+\alpha} E\bigr)$$

holds for all $n \in \mathbb{Z}$.

Proof. (1) $\Longrightarrow$ (2) Use induction on s $(= \dim_A E)$. If $s = 1$, there is nothing to say more. Let $s \geqslant 2$ and assume that our assertion holds for $s - 1$.

By Lemma (3.5) we may further assume that $\operatorname{depth}_A E > 0$. Since E has finite local cohomology, the equality $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ means that $G_\alpha(E)$ also has finite local cohomology. Thus there is an integer $m > 0$ such that $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ (resp. $f_1^{n_1}, f_2^{n_2}, \dots, f_s^{n_s}$) form a u.s.d-sequence on E (resp. $G_\alpha(E)$) for every $n_1, n_2, \dots, n_s \geqslant m$, see Remark (3.1). Hence by Lemma (3.3) and our assumption, we have

$$\left(a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}\right) E \cap \mathfrak{a}^n E = \sum_{i=1}^{s} a_i^{n_i} \mathfrak{a}^{n-n_i} E$$

for all $n \geqslant 0$. By Corollary (2.4), we see $a_1^m E \cap \mathfrak{a}^n E = a_1^m \mathfrak{a}^{n-m} E$. Hence it follows that

$$G_\alpha(E)/f_1^m G_\alpha(E) \cong G_\alpha\left(E/a_1^m E\right) \tag{$\sharp$3.6}$$

as graded R-modules. Because $\operatorname{depth}_A E > 0$ we can see that a_1 must be a non-zero divisor on E, so f_1^m becomes a non-zero divisor on $G_\alpha(E)$. Hence it follows that $\mathrm{H}_N^0(G_\alpha(E)) = (0)$. Thus we have the following.

Claim (3.7). $h^0(G_\alpha(E)) = 0$.

Since both E and $G_\alpha(E)$ are of positive depth, there is two short exact sequences:

$$0 \longrightarrow E \xrightarrow{a_1^m} E \longrightarrow E/a_1^m E \longrightarrow 0, \tag{$\sharp$3.7}$$

$$0 \longrightarrow G_\alpha(E)(-m) \xrightarrow{f_1^m} G_\alpha(E) \longrightarrow G_\alpha\left(E/a_1^m E\right) \longrightarrow 0. \tag{$\sharp$3.8}$$

Recall that a_i^m's (resp. f_i^n's) form a u.s.d-sequence on E (resp. $G_\alpha(E)$). Hence a_1^m (resp. f_1^m) annihilates the local cohomology module $\mathrm{H}_\mathfrak{m}^i(E)$ (resp. $\mathrm{H}_N^i(G_\alpha(E))$) for $i \neq s$, cf. this can be shown in the same way as [S,T]; see also [G-Y] for more explicit statements. From ($\sharp$3.7) and ($\sharp$3.8), we have the following exact sequences:

$$0 \longrightarrow \mathrm{H}_\mathfrak{m}^i(E) \longrightarrow \mathrm{H}_\mathfrak{m}^i\left(E/a_1^m E\right) \longrightarrow \mathrm{H}_\mathfrak{m}^{i+1}(E) \longrightarrow 0, \tag{$\sharp$3.9}$$

$$0 \longrightarrow \mathrm{H}_N^i\left(G_\alpha(E)\right) \longrightarrow \mathrm{H}_N^i\left(G_\alpha\left(E/a_1^m E\right)\right) \longrightarrow \mathrm{H}_N^{i+1}\left(G_\alpha(E)\right)(-m) \longrightarrow 0 \tag{$\sharp$3.10}$$

for each $0 \leqslant i < s-1$. By both ($\sharp$3.9) and ($\sharp$3.10), we know that

$$h^i\left(E/a_1^m E\right) = h^i(E) + h^{i+1}(E) \quad \text{and} \tag{$\sharp$3.11}$$

$$h^i\left(G_\alpha\left(E/a_1^m E\right)\right) = h^i\left(G_\alpha(E)\right) + h^{i+1}\left(G_\alpha(E)\right), \tag{$\sharp$3.12}$$

whence this yields the equality $\mathbb{I}(G_\alpha(E/a_1^m E)) = \alpha \cdot \mathbb{I}(E/a_1^m E)$. Applying the hypothesis of induction on s, we see that $h^i(G_\alpha(E/a_1^m E)) = \alpha \cdot h^i(E/a_1^m E)$ for all $0 \leqslant i < s-1$. Using inductive process on i, by Corollary (2.4) it is easy to check that $h^i(G_\alpha(E)) = \alpha \cdot h^i(E)$ for all $0 < i < s$, cf. ($\sharp$3.11) and ($\sharp$3.12).

The implication (2) $\Longrightarrow$ (1) comes from the definition at once, and this completes the proof of Proposition (3.6). $\square$

Recall that we will say that E is a *quasi-Buchsbaum* A-module, if all local cohomology modules $\mathrm{H}_\mathfrak{m}^i(E)$ ($i \neq s$) of E are annihilated by the maximal ideal $\mathfrak{m}$; i.e., $\mathfrak{m}\mathrm{H}_\mathfrak{m}^i(E) = (0)$ for each $i \neq s$. Moreover, as described in Section 1, we simply say that $G_\alpha(E)$ is quasi-Buchsbaum over R, instead of saying the localization $G_\alpha(E)_N$ by N is so over R_N. Now we are ready to prove Theorem (1.2).

Proof of Theorem (1.2). (1) $\Longrightarrow$ (2) Assume that E is a quasi-Buchsbaum A-module and the equality $\mathbb{I}(G(E)) = \mathbb{I}(E)$ holds. Recall that $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form a weak E-sequence and an unconditioned d-sequence on E for all $n_i \geqslant 2$ ($1 \leqslant i \leqslant s$); cf. see [Su]. Now we shall prove that $f_1^2, f_2^2, \dots, f_s^2$ form a weak $G_\alpha(E)_N$-sequence for each $\alpha > 0$, where $f_i := a_i t$ in R_1 as the above. To prove this it suffices to show that

$$\left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E) : f_i^2 \subseteq \left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E) : N$$

for every $1 \leqslant i \leqslant s$. Let g be a homogeneous element of $G_\alpha(E)$ so that

$$f_i^2 g \in \left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E).$$

Put $n := \deg g$ and choose an element x of $\mathfrak{a}^n E$ such that $g = x \bmod \mathfrak{a}^{n+\alpha}E$ in $[G_\alpha(E)]_n$. By our choice of g, we can express

$$a_i^2 x = y + z$$

with $y \in (a_1^2, \dots, a_{i-1}^2)\mathfrak{a}^n E$ and $z \in \mathfrak{a}^{n+2+\alpha}E$. As z belongs to $(a_1^2, \dots, a_i^2)E \cap \mathfrak{a}^{n+2+\alpha}E$, we see that $z \in (a_1^2, \dots, a_i^2)\mathfrak{a}^{n+\alpha}E$ by Proposition (3.4) and Corollary (2.4). We can find an element v of $\mathfrak{a}^{n+\alpha}E$ so that $a_i^2(x - v) \in (a_1^2, \dots, a_{i-1}^2)E$, hence $x - v \in (a_1^2, \dots, a_{i-1}^2)E : a_i^2$. As a_i^2's form a weak E-sequence we have that

$$\mathfrak{m}(x - v) \subseteq \left(a_1^2, \dots, a_{i-1}^2\right) \cap \mathfrak{a}^n E \subseteq \left(a_1^2, \dots, a_{i-1}^2\right)\mathfrak{a}^{n-2}E,$$
$$\mathfrak{a}(x - v) \subseteq \left(a_1^2, \dots, a_{i-1}^2\right) \cap \mathfrak{a}^{n+1} E \subseteq \left(a_1^2, \dots, a_{i-1}^2\right)\mathfrak{a}^{n-1}E$$

by Proposition (3.4) and Corollary (2.4) again. Since $v \in \mathfrak{a}^{n+\alpha}E$, these observations imply

$$Ng \subseteq \left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E).$$

Hence we know that $G_\alpha(E)$ is a quasi-Buchsbaum R-module.

(2) $\Longrightarrow$ (1) Assume that $G_\alpha(E)$ is a quasi-Buchsbaum R-module for all $\alpha > 0$ and that the equality $\mathbb{I}(G(E)) = \mathbb{I}(E)$ holds. By Proposition (3.4) we know $\mathbb{I}(G_\alpha(E)) = \alpha \cdot \mathbb{I}(E)$ for all $\alpha > 0$. We shall prove that $a_1^2, a_2^2, \dots, a_s^2$ form a weak E-sequence. To prove this we show that

$$\left(a_1^2, \dots, a_{i-1}^2\right)E : a_i^2 \subseteq \left(a_1^2, \dots, a_{i-1}^2\right)E : \mathfrak{m}$$

for every $1 \leqslant i \leqslant s$. Let x be an element of E so that $a_i^2 x \in (a_1^2, \dots, a_{i-1}^2)E$. Put $g := x + \mathfrak{a}^\alpha E$ in $[G_\alpha(E)]_0$. Then we easily see that

$$f_i^2 g \in \left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E).$$

Since f_i^2's form a weak sequence on $G_\alpha(E)_N$ and since the canonical R-homomorphism $W \longrightarrow W_N$ is injective for any graded R-module W, we see that

$$Ng \subseteq \left(f_1^2, \dots, f_{i-1}^2\right)G_\alpha(E). \tag{$\sharp$3.13}$$

Now we further assume that $\alpha \geqslant 3$. Then we have $[G_\alpha(E)]_{-2} = E/\mathfrak{a}^{\alpha-2}E \neq (0)$. Hence looking at the homogeneous component with degree 0 of ($\sharp$3.13) we know that

$$\mathfrak{m}x \subseteq \left(a_1^2, \dots, a_{i-1}^2\right)E + \mathfrak{a}^\alpha E.$$

Applying the Krull's intersection theorem we finally conclude that

$$\mathfrak{m}x \subseteq \bigcap_{\alpha \geqslant 3} (a_1^2, \ldots, a_{i-1}^2)E + \mathfrak{a}^\alpha E = (a_1^2, \ldots, a_{i-1}^2)E.$$

Hence E is a quasi-Buchsbaum A-module. This completes the proof of Theorem (1.2). □

To close this section, we shall discuss about the converse of the assertion (2) of Lemma (3.2). At first, we shall discuss the Cohen–Macaulayness of $\mathrm{Proj}_R\, G_\alpha(E)$. For a graded R-module W, we denote by $\mathrm{Proj}_R\, W$ the set of homogeneous prime ideal $\mathfrak{P}$ of R such that $\mathfrak{P} \in \mathrm{Supp}_R\, W$ and $\mathfrak{P} \not\supseteq R_+$. If the localization $W_\mathfrak{P}$ is a Cohen–Macaulay $R_\mathfrak{P}$-module for all $\mathfrak{P} \in \mathrm{Proj}_R\, W$, we simply say that $\mathrm{Proj}_R\, W$ is Cohen–Macaulay. Then we have the following.

Proposition (3.8). *The following three statements are equivalent.*

(1) $\mathrm{Proj}_R\, G(E)$ *is Cohen–Macaulay.*
(2) $\mathrm{Proj}_R\, G_\alpha(E)$ *is Cohen–Macaulay for all* $\alpha > 0$.
(3) $\mathrm{Proj}_R\, G_\alpha(E)$ *is Cohen–Macaulay for some* $\alpha \geqslant 2$.

Proof. (1) $\Longrightarrow$ (2) Choose any $\mathfrak{P} \in \mathrm{Proj}_R\, G_\alpha(E)$ and localize (♯3.1) at $\mathfrak{P}$. Then we have the following short exact sequence:

$$0 \longrightarrow G(E)_\mathfrak{P} \longrightarrow G_\alpha(E)_\mathfrak{P} \longrightarrow G_{\alpha-1}(E)_\mathfrak{P} \longrightarrow 0.$$

Notice that $\mathfrak{P}$ belongs to both of $\mathrm{Proj}_R\, G(E)$ and $\mathrm{Proj}_R\, G_{\alpha-1}(E)$ by (4) of Lemma (3.2). Using induction on α we may assume that $G_{\alpha-1}(E)_\mathfrak{P}$ and $G(E)_\mathfrak{P}$ too are Cohen–Macaulay. Therefore, $G_\alpha(E)_\mathfrak{P}$ is Cohen–Macaulay. (2) $\Longrightarrow$ (3) Trivial. (3) $\Longrightarrow$ (1) Consider the graded R-submodule $G_\alpha(E)_{\geqslant 0} := \bigoplus_{n \geqslant 0} [G_\alpha(E)]_n$ of $G_\alpha(E)$. There is the following short exact sequence of graded R-modules:

$$0 \longrightarrow G_\alpha(E)_{\geqslant 0} \longrightarrow G_\alpha(E) \longrightarrow G_\alpha(E)/G_\alpha(E)_{\geqslant 0} \longrightarrow 0.$$

Since $G_\alpha(E)/G_\alpha(E)_{\geqslant 0} \cong [G_\alpha(E)]_{1-\alpha} \oplus [G_\alpha(E)]_{2-\alpha} \oplus \cdots \oplus [G_\alpha(E)]_{-1}$, the graded R-module $G_\alpha(E)/G_\alpha(E)_{\geqslant 0}$ is of finite length. It implies

$$G_\alpha(E)_\mathfrak{P} \cong \big(G_\alpha(E)_{\geqslant 0}\big)_\mathfrak{P} \cong R(E)_\mathfrak{P}/\mathfrak{a}^\alpha R(E)_\mathfrak{P}$$

for any homogeneous prime ideal $\mathfrak{P}$ of R such that $\mathfrak{P} \neq N$, where denote by $R(E)$ the Rees module of E with respect to $\mathfrak{a}$, i.e., $R(E) := \bigoplus_{n \geqslant 0} \mathfrak{a}^n E$.

Now let $\mathfrak{P} \in \mathrm{Proj}_R\, G(E)$. Since $G(E) \cong R(E)/\mathfrak{a}R(E)$, $\mathfrak{P}$ contains $\mathfrak{m}$, hence $\mathfrak{a}$, clearly. Since $\mathfrak{P} \not\supseteq R_+$, we can choose an element $a \in \mathfrak{a}$ such that $at \notin \mathfrak{P}$. Put $f := at$. Let b be any element of $\mathfrak{a}$. Since $b = b(f/f) = a(bt/f)$ in R_f, we see $\mathfrak{a}^\alpha R(E)_\mathfrak{P} = a^\alpha \cdot R(E)_\mathfrak{P}$, hence we know

$$G_\alpha(E)_\mathfrak{P} \cong R(E)_\mathfrak{P}/a^\alpha \cdot R(E)_\mathfrak{P}.$$

Since $a \in \mathfrak{a}$ $(\subset \mathfrak{P})$ is a non-zero-divisor on $R(E)_\mathfrak{P}$, the Cohen–Macaulayness of $G_\alpha(E)_\mathfrak{P}$ implies the same one of $R(E)_\mathfrak{P}$. Therefore, $G(E)_\mathfrak{P}$ is also Cohen–Macaulay, because of $G(E)_\mathfrak{P} \cong R(E)_\mathfrak{P}/aR(E)_\mathfrak{P}$. □

Proposition (3.9). *The following three statements are equivalent.*

(1) $G(E)$ *has finite local cohomology.*
(2) $G_\alpha(E)$ *has finite local cohomology for all* $\alpha > 0$.
(3) $G_\alpha(E)$ *has finite local cohomology for some* $\alpha \geqslant 2$.

Proof. (1) $\Longrightarrow$ (2) By (1) of Lemma (3.2) this implication follows at once. (2) $\Longrightarrow$ (3) Trivial. (3) $\Longrightarrow$ (1) Passing to the completion of A, we may assume that A is complete, and hence applying the Cohen's structure theorem we may further assume that A is a homomorphic image of a Cohen–Macaulay local ring. Suppose that $G_\alpha(E)$ has finite local cohomology for some $\alpha \geqslant 2$. By Remark (3.1) it implies that $\mathrm{Proj}_R\, G_\alpha(E)$ is Cohen–Macaulay and $\mathrm{Supp}_R\, G_\alpha(E)$ is equidimensional. Then, by (4) of Lemma (3.2) and Proposition (3.8), it means that $\mathrm{Proj}_R\, G(E)$ is Cohen–Macaulay and $\mathrm{Supp}_R\, G(E)$ is equidimensional. Therefore, by Remark (3.1) again, we finally conclude that $G(E)$ has finite local cohomology. $\square$

4. The Buchsbaumness of $G_\alpha(E)$'s and proof of Theorem (1.1)

This section is devoted to discuss the Buchsbaumness of $G_\alpha(E)$ and to prove our main results, Theorems (1.1) and (1.3).

Throughout we shall keep notations as the same as in the preceding section. Namely, A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and E is a finitely generated A-module of positive dimension s. We always assume that the residue field $A/\mathfrak{m}$ is infinite. Let $\mathfrak{a}$ be a proper ideal of A satisfying $l_A(E/\mathfrak{a}E) < \infty$. Let us still keep the notations $G(E)$, R, N, $\mathbb{I}(E)$ and $G_\alpha(E)$ too. Moreover, setting $\mathfrak{a} = (a_1, a_2, \ldots, a_u)$, the Rees algebra R is usually regarded as the A-subalgebra $A[a_1t, a_2t, \ldots, a_ut]$ of the polynomial ring $A[t]$, where t is an indeterminate over A.

Let us write $u := \mu_A(\mathfrak{a})$ and $v := \mu_A(\mathfrak{m})$, where $\mu_A(*)$ denotes the minimal number of generators of an A-module. Moreover, for a set S we denote by $|S|$ the number of all elements in S. Recall that, for integers i, j, we denote by $[i, j]$ the set of integers n such that $i \leqslant n \leqslant j$. Of course, $[i, j] = \emptyset$ if $i > j$. Then we begin with the following.

Lemma (4.1). *There exist systems of elements in A, say $a_1, a_2, \ldots, a_u$ and $x_1, x_2, \ldots, x_v$, which satisfy the following conditions*:

(1) *$a_1, a_2, \ldots, a_u$ is a minimal system of generators of $\mathfrak{a}$;*
(2) *any s-elements of $a_1t, a_2t, \ldots, a_ut$ in R form a system of parameters for $G(E)$, i.e., there is an integer $r \geqslant 0$ such that $\mathfrak{a}^{r+1}E = (a_i \mid i \in I)\mathfrak{a}^rE$ for all $I \subseteq [1, u]$ with $|I| = s$;*
(3) *$x_1, x_2, \ldots, x_v$ is a minimal system of generators of $\mathfrak{m}$;*
(4) *any s-elements of $a_1, \ldots, a_u, x_1, \ldots, x_v$ form a system of parameters for E.*

Proof. Applying [S-V, Proposition 1.9 in Chap. I], we can find a system of elements of $\mathfrak{a}$, say $a_1, a_2, \ldots, a_u$, such that $a_1t, a_2t, \ldots, a_ut$ form a $G(E)$-basis of the ideal R_+ in the sense of J. Stückrad and W. Vogel, see [S-V, Definition 1.7 in Chap. I] for details. Then according to the method in [S-V, Lemma 2.4 of Chap. IV], this system is extended to the required one immediately, cf. see also [Y1, Remark (5) of §1 (p. 454)]. $\square$

Let $a_1, a_2, \ldots, a_u$ and $x_1, x_2, \ldots, x_v$ be systems of elements in A, which satisfy four conditions given in Lemma (4.1) above. We put $\underline{at} := a_1t, a_2t, \ldots, a_ut$ and $\underline{x} := x_1, x_2, \ldots, x_v$. Let $\mathrm{K}^{\cdot}(\underline{at}, \underline{x}; G_\alpha(E))$ be the Koszul (co-)complex generated (over R) by the system $\underline{at}$, $\underline{x}$ with respect to $G_\alpha(E)$. Since $\underline{at}$, $\underline{x}$ is a minimal system of generators of N, this complex $\mathrm{K}^{\cdot}(\underline{at}, \underline{x}; G_\alpha(E))$ is uniquely determined by the ideal $\mathfrak{N}$ up to isomorphisms not depending on the particular choice of a minimal system of generators, cf. [S-V, §1 of Chapter 0 (p. 27)]. Hence we denote it by $\mathrm{K}^{\cdot}(N; G_\alpha(E))$ simply. Notice that the Koszul complex $\mathrm{K}^{\cdot}(N; G_\alpha(E))$ is a complex of direct sums of copies of a graded module $G_\alpha(E)$. Hence we have an expression of it as follows:

$$\mathrm{K}^{\cdot}\big(N; G_\alpha(E)\big) = \bigoplus_{\substack{I \subseteq [1,u] \\ J \subseteq [1,v]}} G_\alpha(E) \cdot e_J^I, \qquad \mathrm{K}^i\big(N; G_\alpha(E)\big) = \bigoplus_{|I|+|J|=i} G_\alpha(E) \cdot e_J^I$$

where $\{e_J^I \mid I \subseteq [1, u],\ J \subseteq [1, v]\}$ is the graded free basis with $\deg e_J^I = -|I|$.

With these notations, we shall prove Theorem (1.1).

Proof of Theorem (1.1). Suppose that E is a Buchsbaum A-module of dimension $s > 0$ and the equality $\mathbb{I}(G(E)) = \mathbb{I}(E)$ holds. By induction on s we shall show the Buchsbaumness of $G_\alpha(E)$ over R. Notice that $G_\alpha(E)$ is quasi-Buchsbaum by Theorem (1.2). There is nothing to say any more for $s = 1$. We may assume that $s \geqslant 2$ and our assertion is true for $s - 1$. In order to get our assertion, it is enough to show the canonical map

$$\phi^i_{G_\alpha(E)} : \mathrm{H}^i\big(N; G_\alpha(E)\big) \longrightarrow \varinjlim_n \mathrm{H}^i\big(\underline{at}(n), \underline{x}(n); G_\alpha(E)\big) \cong \mathrm{H}^i_N\big(G_\alpha(E)\big) \qquad (\sharp 4.1)$$

is surjective for all $0 \leqslant i < s$, where $\underline{at}(n) := (a_1t)^n, (a_2t)^n, \ldots, (a_ut)^n$, and $\underline{x}(n) := x_1^n, x_2^n, \ldots, x_v^n$; see [S-V, Theorem 2.15 in Chap. I].

First, we consider the case where $\operatorname{depth}_A E > 0$. Put $a := a_1$ and $f := a_1t$. Because $\operatorname{depth}_A E > 0$ we can see that a_1 must be a non-zero divisor on E, and then this linear element $f \in R_1$ becomes a non-zero-divisor on $G_\alpha(E)$. Hence, there is a short exact sequence of graded R-modules

$$0 \longrightarrow G_\alpha(E)(-1) \xrightarrow{f} G_\alpha(E) \longrightarrow G_\alpha(E/aE) \longrightarrow 0,$$

because $G_\alpha(E)/f \cdot G_\alpha(E) \cong G_\alpha(E/aE)$ holds by Corollary (2.4). By Proposition (3.4), we can easily check that the equality $\mathbb{I}(G(E/aE)) = \mathbb{I}(E/aE)$ holds. Applying the hypothesis of induction to E/aE, we see that $G_\alpha(E/aE)$ is a Buchsbaum module over R. Look at the following commutative diagram with exact rows:

$$\begin{CD}
\mathrm{H}^{i-1}(N; G_\alpha(E/aE)) @>>> \mathrm{H}^i(N; G_\alpha(E))(-1) @>f>> 0\\
@V{\phi^{i-1}_{G_\alpha(E/aE)}}VV @VV{\phi^i_{G_\alpha(E)}(-1)}V \\
\mathrm{H}^{i-1}_N(G_\alpha(E/aE)) @>>> \mathrm{H}^i_N(G_\alpha(E))(-1) @>f>> 0.
\end{CD}$$

The surjectivity of $\phi^{i-1}_{G_\alpha(E/aE)}$ implies the same one of $\phi^i_{G_\alpha(E)}$ for $0 < i < s$, thus $G_\alpha(E)$ is a Buchsbaum module over R.

Next, we shall deal with the case where $\operatorname{depth}_A E = 0$, hence $\mathrm{H}^0_N(G_\alpha(E)) \neq (0)$. By ($\sharp$3.4) and Lemma (3.5), we have the short exact sequence of graded R-modules as follows:

$$0 \longrightarrow \mathrm{H}^0_N\big(G_\alpha(E)\big) \longrightarrow G_\alpha(E) \longrightarrow G_\alpha(E/U) \longrightarrow 0,$$

where $U := \mathrm{H}^0_{\mathfrak{m}}(E)$. By the argument as in the case of positive depth, we already know that $G_\alpha(E/U)$ is a Buchsbaum module over R. So $\phi^i_{G_\alpha(E/U)}$ is surjective for all $i \neq s$. According to the proof of [S-V, Theorem 2.15 in Chap. I], the surjectivity of $\phi^i_{G_\alpha(E)}$ is coming from the injectivity of the canonical map

$$\tau^i_E : \mathrm{H}^i\big(N; \mathrm{H}^0_N\big(G_\alpha(E)\big)\big) \longrightarrow \mathrm{H}^i\big(N; G_\alpha(E)\big) \qquad (\sharp 4.2)$$

for all $0 \leqslant i \leqslant s$. For the case where $0 \leqslant i \leqslant s - 1$ this is obvious. In fact, since f is a d-sequence on $G_\alpha(E)$ by Proposition (3.4), we have $\mathrm{H}^0_N(G_\alpha(E)) \cap f \cdot G_\alpha(E) = (0)$. So there is a commutative diagram

$$\begin{CD}
\mathrm{H}^0_N(G_\alpha(E)) @>{\tau_E}>> G_\alpha(E)\\
@V{\sigma}VV @VVV\\
\mathrm{H}^0_{\mathfrak{N}}(G_\alpha(E/aE)) @>>{\tau_{E/aE}}> G_\alpha(E/aE)
\end{CD}$$

such that $\tau \circ \sigma$ is injective. Since $\mathrm{H}^0_N(G_\alpha(E))$ and $\mathrm{H}^0_N(G_\alpha(E/\mathfrak{a}E))$ are vector spaces over R/N, the vertical map σ must split. Consequently, the induced map by σ given in the following commutative diagram, say σ^i, splits too:

$$\begin{tikzcd}
\mathrm{H}^i(N; \mathrm{H}^0_N(G_\alpha(E))) \arrow[r, "\tau^i_E"] \arrow[d, "\sigma^i"'] & \mathrm{H}^i(N; G_\alpha(E)) \arrow[d] \\
\mathrm{H}^i(N; \mathrm{H}^0_N(G_\alpha(E/\mathfrak{a}E))) \arrow[r, "\tau^i_{E/\mathfrak{a}E}"] & \mathrm{H}^i(N; G_\alpha(E/\mathfrak{a}E)).
\end{tikzcd}$$

By induction hypothesis on s, the map $\tau^i_{E/\mathfrak{a}E}$ is injective for $0 \leqslant i \leqslant s-1$, hence τ^i_E is so. Therefore it remains to show the injectivity of the map τ^s_E:

$$\tau^s_E : \mathrm{H}^s\big(N; \mathrm{H}^0_N\big(G_\alpha(E)\big)\big) \longrightarrow \mathrm{H}^s\big(N; G_\alpha(E)\big). \qquad \square \tag{$\sharp$4.3}$$

Though the injectivity of ($\sharp$4.3) is coming from similar arguments as in the proof of [Y1, Theorem 4.1], we shall here discuss it quickly for reader's convenience. Via

$$0 \longrightarrow \mathrm{H}^0_N\big(G_\alpha(E)\big) \longrightarrow G_\alpha(E) \longrightarrow G_\alpha(E/U) \longrightarrow 0,$$

we have an exact sequence of graded Koszul complexes as follows:

$$0 \longrightarrow \mathrm{K}^{\cdot}\big(N; \mathrm{H}^0_N\big(G_\alpha(E)\big)\big) \longrightarrow \mathrm{K}^{\cdot}\big(N; G_\alpha(E)\big) \longrightarrow \mathrm{K}^{\cdot}\big(N; G_\alpha(E/U)\big) \longrightarrow 0.$$

Since $[\mathrm{H}^0_N(G_\alpha(E))]_n = U \cap \mathfrak{a}^n E / U \cap \mathfrak{a}^{n+\alpha} E$ by Proposition (3.6) and since $\mathfrak{m}U = (0)$, it is easy to see

$$N \cdot \mathrm{H}^0_N\big(G_\alpha(E)\big) = (0). \tag{$\sharp$4.4}$$

Thus all of the differentiations of the Koszul complex $\mathrm{K}^{\cdot}(N; \mathrm{H}^0_N(G_\alpha(E)))$ are zero maps. To show the injectivity of the map ($\sharp$4.3), we look for the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}^{s-1}(N; \mathrm{H}^0_N(G_\alpha(E))) \arrow[r] \arrow[d, "0"'] & \mathrm{K}^{s-1}(N; G_\alpha(E)) \arrow[d, "\partial"] \\
0 \arrow[r] & \mathrm{K}^{s}(N; \mathrm{H}^0_N(G_\alpha(E))) \arrow[r] & \mathrm{K}^{s}(N; G_\alpha(E)).
\end{tikzcd}$$

Choose a homogeneous element $\xi \in \mathrm{K}^s(N; \mathrm{H}^0_N(G_\alpha(E)))$ with $\deg \xi = n$, where $n \in \mathbb{Z}$, and assume that there exists a homogeneous element $\eta \in \mathrm{K}^{s-1}(N; G_\alpha(E))$ such that $\xi = \partial(\eta)$ in $\mathrm{K}^s(N; G_\alpha(E))$. Our goal is to prove $\xi = 0$. So write ξ, η as follows:

$$\xi = \sum_{|I|+|J|=s} \xi^I_J \cdot e^I_J, \qquad \eta = \sum_{|P|+|Q|=s-1} \eta^P_Q \cdot e^P_Q,$$

where $\xi^I_J \in [G_\alpha(E)]_{n+|I|}$ and $\eta^P_Q \in [G_\alpha(E)]_{n+|P|}$. Then we have

$$\xi^I_J = \sum_{j \in J} (-1)^{J(j)} x_j \cdot \eta^I_{J\setminus\{j\}} + \sum_{i \in I} (-1)^{|J|+I(i)} a_i t \cdot \eta^{I\setminus\{i\}}_J,$$

here the notation $J(j)$ denotes the number of elements of the set $\{j' \in J \mid j' < j\}$, i.e., $J(j) := |\{j' \in J \mid j' < j\}|$, and $I(i)$ is defined in the same meaning too. Since $\xi \in \mathrm{K}^s(N; \mathrm{H}^0_N(G_\alpha(E)))$ this is equivalent to say that $\xi^I_J \in \mathrm{H}^0_N(G_\alpha(E))$ for all I, J, and so we shall prove that $\xi^I_J = 0$ in $G_\alpha(E)$ for all I, J with $|I| + |J| = s$.

Choose a representation of η^P_Q, say c^P_Q, i.e., c^P_Q is an element of $\mathfrak{a}^{n+|P|}E$ such that $\overline{c^P_Q} = \eta^P_Q$ in $[G_\alpha(E)]_{n+|P|}$, here we denote by $\bar{c}$ the homogeneous element $c \bmod \mathfrak{a}^{m+1}E$ in $[G_\alpha(E)]_m$ for each $c \in \mathfrak{a}^m E$. Using these elements c^P_Q's, we define a representation of the element ξ^I_J, say b^I_J, as follows:

$$b^I_J := \sum_{j \in J} (-1)^{J(j)} x_j \cdot c^I_{J \setminus \{j\}} + \sum_{i \in I} (-1)^{|J| + I(i)} a_i \cdot c^{I \setminus \{i\}}_J. \tag{♯4.5}$$

Then it is easy to see that $b^I_J \in \mathfrak{a}^{n+|I|}E$ and $\overline{b^I_J} = \xi^I_J$ in $[G_\alpha(E)]_{n+|I|}$. Notice that, however, this representation b^I_J of ξ^I_J depends on a choice of representations c^P_Q's.

To get $\xi^I_J = 0$ in $G_\alpha(E)$ it is enough to show $b^I_J = 0$ in E. But thus is not true in general. Under a special situation, however, we can conclude that $b^I_J = 0$ in E. Namely we have the following, which is a key lemma in our argument.

Lemma (4.2). *Suppose that there exists another subset I' of $[1, u]$ which satisfies the following two conditions*:

(i) $I' \supset I$ *and* $|I'| = |I| + 1$,
(ii) $b^{I'}_{J'} = 0$ *in* E *for all* $J' \subset J$ *such that* $|J'| = |J| - 1$.

Then, after a suitable change of the representations $c^{I \setminus \{i\}}_J$'s ($i \in I$), it follows that $b^I_J = 0$ in E. (Notice that, in the case $|I| = s$ there is nothing to say about the hypothesis, and in the case $I = \emptyset$ the assertion $b^{\emptyset}_J = 0$ in E follows under no change of the representations.)

Proof. First of all, we deal with the case $|I| = s$. Then $J = \emptyset$ clearly, and we have

$$\xi^I_\emptyset = \sum_{i \in I} (-1)^{I(i)} a_i t \cdot \eta^{I \setminus \{i\}}_\emptyset.$$

This implies

$$\xi^I_\emptyset \in (a_i t \mid i \in I) \cdot G_\alpha(E) \cap \mathrm{H}^0_N\big(G_\alpha(E)\big) = (0)$$

because $a_i t$'s from a d-sequence on $G_\alpha(E)$ by Proposition (3.4). Thus

$$\sum_{i \in I} (-1)^{I(i)} a_i c^{I \setminus \{i\}}_\emptyset \in (a_i \mid i \in I)E \cap \mathfrak{a}^{n+s+\alpha} E = (a_i \mid i \in I)\mathfrak{a}^{n+s-1+\alpha} E$$

cf. Corollary (2.4). So, we can find elements g_i's of $\mathfrak{a}^{n+s+\alpha-1}E$ such that

$$\sum_{i \in I} (-1)^{I(i)} a_i c^{I \setminus \{i\}}_\emptyset = \sum_{i \in I} (-1)^{I(i)} a_i g_i$$

in E. Since $g_i \in \mathfrak{a}^{n+s-1+\alpha} E$ we have

$$\eta^{I \setminus \{i\}}_\emptyset = \overline{c^{I \setminus \{i\}}_\emptyset} = \overline{c^{I \setminus \{i\}}_\emptyset - g_i},$$

because of $\eta_\emptyset^{I\setminus\{i\}} \in [G_\alpha(E)]_{n+s-1} = \mathfrak{a}^{n+s-1}E/\mathfrak{a}^{n+s-1+\alpha}E$. Therefore, after changing a representation $c_\emptyset^{I\setminus\{i\}}$ of $\eta_\emptyset^{I\setminus\{i\}}$ with the element $c_\emptyset^{I\setminus\{i\}} - g_i$, we obtain that

$$b_\emptyset^I = \sum_{i\in I}(-1)^{I(i)}a_i c_\emptyset^{I\setminus\{i\}} = 0.$$

Next, let us consider the case that $|I| < s$. Then $J \neq \emptyset$ clearly, and we claim the following.

Claim (4.3). $\sum_{j\in J}(-1)^{J(j)}x_j c^I_{J\setminus\{j\}} \in (a_i \mid i \in I)E$.

Proof. Let $j \in J$. Applying the assumption (ii) of our Lemma (4.2) to each $J \setminus \{j\}$ we have

$$0 = b^{I'}_{J\setminus\{j\}} = \sum_{j'\in J\setminus\{j\}}(-1)^{J\setminus\{j\}(j')}x_{j'}c^{I'}_{J\setminus\{j,j'\}} + \sum_{i'\in I'}(-1)^{|J|-1+I'(i')}a_{i'}c^{I'\setminus\{i'\}}_{J\setminus\{j\}}.$$

Multiplying by $(-1)^{J(j)}x_j$ and taking the sum $\sum_{j\in J}$, we get that

$$\begin{aligned} 0 &= \sum_{j\in J}\sum_{j'\in J\setminus\{j\}}(-1)^{J(j)}(-1)^{J\setminus\{j\}(j')}x_j x_{j'}c^{I'}_{J\setminus\{j,j'\}} + \sum_{i'\in I'}(-1)^{|J|-1+I'(i')}a_{i'}\left(\sum_{j\in J}(-1)^{J(j)}x_j c^{I'\setminus\{i'\}}_{J\setminus\{j\}}\right) \\ &= \sum_{i'\in I'}(-1)^{|J|-1+I'(i')}a_{i'}\left(\sum_{j\in J}(-1)^{J(j)}x_j c^{I'\setminus\{i'\}}_{J\setminus\{j\}}\right) \end{aligned}$$

in E, because of $\sum_j\sum_{j'}\pm x_j x_{j'}c^{I'}_{J\setminus\{j,j'\}} = 0$. Choose an element $i'' \in I' \setminus I$. Then $I = I' \setminus \{i''\}$. Notice that a_i's and x_j's form a system of parameters for E by our choices (recall that $|I| + |J| = s$), and $(a_i \mid i \in I)E : a_{i''} = (a_i \mid i \in I)E : \mathfrak{m} \subseteq (a_i \mid i \in I)E : x_{j'}$ holds for any $j' \in J$. By these observations we get that

$$\begin{aligned} \sum_{j\in J}(-1)^{J(j)}x_j c^I_{J\setminus\{j\}} &\in \left[(a_i \mid i \in I)E : a_{i''}\right] \cap (x_j \mid j \in J)E \\ &\subseteq \left[(a_i \mid i \in I)E : x_{j'}\right] \cap (a_i, x_j \mid i \in I,\ j \in J)E \\ &= (a_i \mid i \in I)E, \end{aligned}$$

and this is finished the proof of Claim (4.3). □

Now we shall continue the proof of Lemma (4.2). Since $c^I_{J\setminus\{j\}} \in \mathfrak{a}^{n+|I|}E$, we see by Claim (4.3) that

$$\sum_{j\in J}(-1)^{J(j)}x_j c^I_{J\setminus\{j\}} \in (a_i \mid i \in I)E \cap \mathfrak{a}^{n+|I|}E = (a_i \mid i \in I)\mathfrak{a}^{n+|I|-1}E.$$

This means

$$\xi^I_J = \overline{\sum_{j\in J}(-1)^{J(j)}x_j c^I_{J\setminus\{j\}} + \sum_{i\in I}(-1)^{|J|+I(i)}a_i c^{I\setminus\{i\}}_J} \in (a_i t \mid i \in I)\cdot G_\alpha(E).$$

Since $\xi^I_J \in \mathrm{H}^0_N(G_\alpha(E))$ we see $\xi^I_J = 0$ by the d-sequence property of $a_i t$'s again. Hence we get

$$b_J^I = \sum_{j\in J}(-1)^{J(j)}x_j c_{J\setminus\{j\}}^I + \sum_{i\in I}(-1)^{|J|+I(i)}a_i c_J^{I\setminus\{i\}}$$
$$\in (a_i \mid i\in I)E\cap \mathfrak{a}^{n+|I|+\alpha}E = (a_i \mid i\in I)\mathfrak{a}^{n+|I|-1+\alpha}E$$

in E. Recall that $c_J^{I\setminus\{i\}}$ is a representation of $\eta_J^{I\setminus\{i\}} \in [G_\alpha(E)]_{n+|I|-1}$. Therefore, after a suitable change representations $c_J^{I\setminus\{i\}}$'s $(i\in I)$ in the same way at the beginning, we finally conclude that $b_J^I = 0$ in E. This completes the proof of Lemma (4.2). □

Now we continue our proof of Theorem (1.1).

Proof of Theorem (1.1) (continued). Let ξ_J^I, c_Q^P and b_J^I be the same notations as above, cf. recall (♯4.5) for the definition. In order to get $\xi_J^I = 0$ in $G_\alpha(E)$ for all $I\subseteq[1,u]$, $J\subseteq[1,v]$ with $|I|+|J|=s$, it is enough to show that $b_J^I = 0$ in E, after a suitable change of the representations c_Q^P's.

Take $I\subseteq[1,u]$, $J\subseteq[1,v]$ such that $|I|+|J|=s$, and fix them. Write $k:=|I|$. Then $0\leqslant k\leqslant s$ and $|J|=s-k$ clearly. After relabeling among the system $a_1,a_2,\dots,a_u$ we may assume that $I=[1,k]$. Let us introduce one more useful notation, say Φ_l for each $k\leqslant l\leqslant s$, as follows:

$$\Phi_l := \bigl\{\bigl([1,l]\setminus\{i\},Q\bigr)\bigm| 1\leqslant i\leqslant l,\ Q\subseteq J \text{ with } |Q|=s-l\bigr\}.$$

Then it is easy to see that $\Phi_l\cap\Phi_{l'}=\emptyset$ for all $l\neq l'$.

Using descending induction on k, we shall show that $b_{J'}^{[1,l]}=0$ in E for $J'\subseteq J$ with $|J'|=s-l$ and $k\leqslant l\leqslant s$, after a suitable change of the representations c_Q^P's for $(P,Q)\in\bigcup_{l=k}^s\Phi_l$. If $k=s$, our assertion is obvious by Lemma (4.2). Now let $0\leqslant k<s$, and assume that we have already found the representations c_Q^P's for $(P,Q)\in\bigcup_{l=k+1}^s\Phi_l$ such that $b_{J'}^{[1,l]}=0$ in E for $J'\subseteq J$ with $|J'|=s-l$ and $k+1\leqslant l\leqslant s$. Then applying Lemma (4.2) again to the sets $[1,k+1]\supset[1,k]$, we can suitably change the representations c_Q^P's for $(P,Q)\in\Phi_k$ such that $b_{J'}^{[1,k]}=0$ in E for $J'\subseteq J$ with $|J'|=s-k$. Finally it follows that $b_{J'}^{[1,l]}=0$ in E for $J'\subseteq J$ with $|J'|=s-l$ and $k\leqslant l\leqslant s$, after a suitable change of the representations c_Q^P's for $(P,Q)\in\bigcup_{l=k}^s\Phi_l$. This finishes the proof of Theorem (1.1). □

To close this section, we shall prove the rest of our main results Theorem (1.3). From now on, let E be a finitely generated A-module of dimension $s>0$. Then:

Proof of Theorem (1.3). Suppose that E has finite local cohomology and the equality $\mathbb{I}(G(\mathfrak{m},E))=\mathbb{I}(E)$ holds. By Theorem (1.1) it is enough to show (2) $\Longrightarrow$ (1). Assume that $G_\alpha(\mathfrak{m},E)$ is a Buchsbaum $R(\mathfrak{m})$-module for all $\alpha>0$. By Theorem (1.2), E is quasi-Buchsbaum over A. There is nothing to say any more in the case where $s=1$. So we may further assume that $s\geqslant 2$ and our assertion is true for $s-1$.

Let $1\leqslant i\leqslant u$. Then we claim the following.

Claim (4.4). *$a_iE\cap\mathfrak{m}^nE=a_i\mathfrak{m}^{n-1}E$ holds for all $n\in\mathbb{Z}$.*

Proof. Choose an element $x\in E$ such that $a_ix\in a_iE\cap\mathfrak{m}^nE$. Since E is quasi-Buchsbaum we see $a_i^2E\cap\mathfrak{m}^{n+1}E=a_i^2\mathfrak{m}^{n-1}E$ by Corollary (2.4) and Proposition (3.4). Then $a_i^2x=a_i^2y$ for some $y\in\mathfrak{m}^{n-1}E$. Since $a_1^2,a_2^2,\dots,a_s^2$ form a weak E-sequence in any order, we know

$$x-y\in 0:a_i^2=0:\mathfrak{m}\subseteq 0:a_i.$$

Hence we finally get that $a_ix=a_iy\in a_i\mathfrak{m}^{n-1}E$. □

Claim (4.4) means that

$$G_\alpha(\mathfrak{m}, E)/f_i \cdot G_\alpha(\mathfrak{m}, E) \cong G_\alpha(\mathfrak{m}, E/a_iE). \tag{$\sharp$4.6}$$

This leads the next exact sequence of graded $R(\mathfrak{m})$-modules:

$$0 \longrightarrow \mathrm{H}^0_N\big(G_\alpha(\mathfrak{m}, E)\big)(-1) \longrightarrow G_\alpha(\mathfrak{m}, E)(-1) \xrightarrow{f_i} G_\alpha(\mathfrak{m}, E) \longrightarrow G_\alpha(\mathfrak{m}, E/a_iE) \longrightarrow 0.$$

By ($\sharp$4.6) and Proposition (3.4) and by the quasi-Buchsbaumness of both $G_\alpha(\mathfrak{m}, E)$ and E, it follows that

$$\mathbb{I}\big(G_\alpha(\mathfrak{m}, E/a_iE)\big) = \mathbb{I}\big(G_\alpha(\mathfrak{m}, E)\big) = \alpha\mathbb{I}(E) = \alpha\mathbb{I}(E/a_iE).$$

In particular, we have $\mathbb{I}(G(\mathfrak{m}, E/a_iE)) = \mathbb{I}(E/a_iE)$ from the case where $\alpha = 1$. Furthermore, by ($\sharp$4.6), we obviously know that $G_\alpha(\mathfrak{m}, E/a_iE)$ is a Buchsbaum $R(\mathfrak{m})$-module for all α. Applying the hypothesis of induction on s, we know that E/a_iE is Buchsbaum over A for each $1 \leqslant i \leqslant u$.

According to [G3, Proposition (2.12)], we conclude that E is a Buchsbaum A-module. However we shall here give much simpler proof. First, suppose $\operatorname{depth}_A E > 0$. Then we look for the following commutative diagram of A-modules:

$$\begin{tikzcd}
\mathrm{H}^{i-1}(\mathfrak{m}; E/aE) \arrow[r] \arrow[d, "\phi^{i-1}_{E/aE}"'] & \mathrm{H}^i(\mathfrak{m}; E) \arrow[r, "a"] \arrow[d, "\phi^i_E"] & 0 \\
\mathrm{H}^{i-1}_\mathfrak{m}(E/aE) \arrow[r] & \mathrm{H}^i_\mathfrak{m}(E) \arrow[r, "a"] & 0
\end{tikzcd}$$

for $1 \leqslant i < s$, where we put $a := a_1$. Thus the surjectivity of $\phi^{i-1}_{E/aE}$ implies the same one of ϕ^i_E, namely E is Buchsbaum over A in this case. Next, we deal with the case where $\operatorname{depth}_A E = 0$. It is enough to show that the canonical map

$$\mathrm{H}^i\big(\mathfrak{m}; \mathrm{H}^0_\mathfrak{m}(E)\big) \longrightarrow \mathrm{H}^i(\mathfrak{m}; E)$$

is injective for all $0 \leqslant i \leqslant s$. This injectivity can be easily shown from the following:

$$(a_i \mid i \in I)E \cap \mathrm{H}^0_\mathfrak{m}(E) = (0)$$

for any $I \subseteq [1, u]$ with $|I| = s$, because of $\mathfrak{m} = (a_1, a_2, \ldots, a_u)$. To show this, we may assume $I = [1, s]$ without loss of the generality of our arguments. Choose an element $x \in (a_1, a_2, \ldots, a_s)E \cap \mathrm{H}^0_\mathfrak{m}(E)$ and express

$$x = a_1x_1 + a_2x_2 + \cdots + a_sx_s$$

with $x_i \in E$. Put $g := x + \mathfrak{m}^\alpha E \in [G_\alpha(\mathfrak{m}, E)]_0$ and $g_i := x_i + \mathfrak{m}^{\alpha-1}E \in [G_\alpha(\mathfrak{m}, E)]_{-1}$ for any $\alpha \geqslant 2$. Then

$$g = f_1g_1 + f_2g_2 + \cdots + f_sg_s,$$

clearly. Since $G_\alpha(\mathfrak{m}, E)$ is a Buchsbaum $R(\mathfrak{m})$-module, $f_1, f_2, \ldots, f_s$ form a d-sequence on $G_\alpha(\mathfrak{m}, E)$, hence we have $g = 0$ in $[G_\alpha(E)]_0$. This implies $x \in \mathfrak{m}^\alpha E$, hence

$$x \in \bigcap_{\alpha \geqslant 2} \mathfrak{m}^\alpha E = (0)$$

by the Krull's intersection theorem. This completes the proof of Theorem (1.3). $\square$

5. Example and remark

There is a Noetherian local ring $(A, \mathfrak{m})$ such that $G_\alpha(\mathfrak{m})$ is a Buchsbaum module over $R(\mathfrak{m})$ in a suitable range of the integer α, but A itself is not a Buchsbaum ring.

Example (5.1). Let $S := k[\![X, Y, Z, W_1, \ldots, W_{d-1}]\!]$ be a formal power series ring over a field k, and put

$$A := S/\big(X^2, XY, XZ - Y^r, XZ^2, XW_1, \ldots, XW_{d-1}\big),$$

where $d > 0$ and $r \geqslant 3$ are integers: cf. [G3, Example (4.10)]. Moreover, we denote by N the unique homogeneous maximal ideal of the Rees algebra $R(\mathfrak{m})$; i.e., $N := \mathfrak{m}R(\mathfrak{m}) + R(\mathfrak{m})_+$. Then we have the following.

(i) $A/\mathrm{H}^0_\mathfrak{m}(A)$ is a Cohen–Macaulay ring and $\mathfrak{m}\mathrm{H}^0_\mathfrak{m}(A) \neq (0)$. Hence A has finite local cohomology, but it is not a (quasi-)Buchsbaum ring.
(ii) $G(\mathfrak{m})$ is a Buchsbaum ring and $\mathbb{I}(G(\mathfrak{m})) = \mathbb{I}(A) = 2$.
(iii) $G_\alpha(\mathfrak{m})/\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))$ is a Cohen–Macaulay module over $R(\mathfrak{m})$ for all $\alpha > 0$.
(iv) $G_\alpha(\mathfrak{m})$ is a Buchsbaum module over $R(\mathfrak{m})$ if and only if $\alpha \leqslant r - 2$.

In fact, put $U := \mathrm{H}^0_\mathfrak{m}(A)$ and $\overline{A} := A/U$. We denote by x, y, z the images of indeterminates X, Y, Z via the canonical projection $S \longrightarrow A$, respectively. It is easy to see that $A/(x) \cong S/(X, Y^r) \cong k[\![Y, Z]\!]/(Y^r)$ as k-algebras, and $\mathfrak{m}x = (xz) = (y^r) \neq (0)$ in A, also $\mathfrak{m}^2 x = (0)$. This means $U = (x)$, hence $\overline{A}$ is a Cohen–Macaulay ring (of dimension d). Moreover, we know that $l_A(U) = 2$, hence we have a chain of A-submodule of U as follows:

$$U = U \cap \mathfrak{m} \supsetneq U \cap \mathfrak{m}^2 = \cdots = U \cap \mathfrak{m}^r = \mathfrak{m}U \supsetneq U \cap \mathfrak{m}^{r+1} = (0). \tag{$\sharp$5.1}$$

Thus, A has finite local cohomology with $\mathbb{I}(A) = 2$, but A itself is not a (quasi-)Buchsbaum ring. Now let us consider the associated graded ring $G(\mathfrak{m})$ of $\mathfrak{m}$. Denote by $\overline{\mathfrak{m}}$ the maximal ideal of $\overline{A}$. Let $\psi : G(\mathfrak{m}) \longrightarrow G(\overline{\mathfrak{m}})$ be the canonical epimorphism and put $U^* := \operatorname{Ker}\psi$. Since $\overline{A} \cong S/(X, Y^r)$, we can easily check that $G(\overline{\mathfrak{m}}) \cong k[X, Y, Z, W_1, \ldots, W_{d-1}]/(X, Y^r)$, as graded k-algebras. This implies that $G(\overline{\mathfrak{m}})$ must be a Cohen–Macaulay ring. Using Lemma (3.5) and ($\sharp$5.1), we can easily check that

$$\big[\mathrm{H}^0_N\big(G(\mathfrak{m})\big)\big]_n = \begin{cases} k & (n = 1, r), \\ (0) & (\text{else}). \end{cases} \tag{$\sharp$5.2}$$

This means that the equality $\mathbb{I}(G(\mathfrak{m})) = 2 = \mathbb{I}(A)$ holds and that $\mathrm{H}^0_N(G(\mathfrak{m}))$ have to be annihilated by N (recall $r \geqslant 3$). Hence $G(\mathfrak{m})$ becomes a Buchsbaum ring.

Since $\mathbb{I}(G(\mathfrak{m})) = \mathbb{I}(A)$, we see that $\mathbb{I}(G_\alpha(\mathfrak{m})) = \alpha \cdot \mathbb{I}(A)$ for all $\alpha > 0$ by Proposition (3.4). Moreover, by Proposition (3.6) it follows that

$$\big[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\big]_n = \big(U \cap \mathfrak{m}^n\big)/\big(U \cap \mathfrak{m}^{n+\alpha}\big) \tag{$\sharp$5.3}$$

for all $\alpha > 0$ and $n \in \mathbb{Z}$, and

$$G_\alpha(\mathfrak{m})/\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big) \cong G_\alpha(\mathfrak{m}, \overline{A}) \cong G_\alpha(\overline{\mathfrak{m}})$$

for all $\alpha > 0$. Thus we get $\mathbb{I}(G_\alpha(\mathfrak{m})/\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))) = \mathbb{I}(G_\alpha(\overline{\mathfrak{m}})) = \alpha \cdot \mathbb{I}(\overline{A}) = 0$ by Proposition (3.6) again, namely $G_\alpha(\mathfrak{m})/\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))$ is a Cohen–Macaulay module over $R(\mathfrak{m})$. By ($\sharp$5.1) and ($\sharp$5.3) we know

$$\big[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\big]_n = \begin{cases} U/\mathfrak{m}U & (2 - \alpha \leqslant n \leqslant 1), \\ \mathfrak{m}U & (r - \alpha < n \leqslant r), \\ (0) & (\text{else}) \end{cases} \tag{$\sharp$5.4}$$

if $r-\alpha \geqslant 1$, and

$$\left[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\right]_n = \begin{cases} U/\mathfrak{m}U & (2-\alpha \leqslant n \leqslant r-\alpha), \\ U & (r-\alpha < n \leqslant 1), \\ \mathfrak{m}U & (2 \leqslant n \leqslant r), \\ (0) & (\text{else}) \end{cases} \tag{$\sharp$5.5}$$

if $r-\alpha \leqslant 0$.

If $r-\alpha \geqslant 2$, then we see

$$\mathfrak{m}t \cdot \left[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\right]_1 \subseteq \left[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\right]_2 = (0)$$

by ($\sharp$5.4). So we get $N \cdot \mathrm{H}^0_N(G_\alpha(\mathfrak{m})) = (0)$. Thus $G_\alpha(\mathfrak{m})$ is a Buchsbaum module over $R(\mathfrak{m})$. Now, let us consider the case where $r-\alpha \leqslant 1$. At first, if $r-\alpha = 1$, then we have $[\mathrm{H}^0_N(G_{r-1}(\mathfrak{m}))]_1 = U/\mathfrak{m}U$ and $[\mathrm{H}^0_N(G_{r-1}(\mathfrak{m}))]_2 = \mathfrak{m}U$ by ($\sharp$5.4), so it is easy to see that

$$\mathfrak{m}t \cdot \left[\mathrm{H}^0_N\big(G_{r-1}(\mathfrak{m})\big)\right]_1 = \mathfrak{m}U \neq (0)$$

in $[\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))]_2$. Next, if $r-\alpha \leqslant 0$, then we have $[\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))]_1 = U$ by ($\sharp$5.5), hence

$$\mathfrak{m} \cdot \left[\mathrm{H}^0_N\big(G_\alpha(\mathfrak{m})\big)\right]_0 = \mathfrak{m}U \neq (0)$$

in $[\mathrm{H}^0_N(G_\alpha(\mathfrak{m}))]_0$. These observations imply that $N \cdot \mathrm{H}^0_N(G_\alpha(\mathfrak{m})) \neq (0)$, namely, $G_\alpha(\mathfrak{m})$ is not a Buchsbaum module over $R(\mathfrak{m})$ in the case where $r-\alpha \leqslant 1$. Therefore, $G_\alpha(\mathfrak{m})$ is a Buchsbaum module over $R(\mathfrak{m})$ if and only if $r-\alpha \geqslant 2$, i.e., $\alpha \leqslant r-2$.

Remark (5.2). Here let us reformulate the statement (iv) of [Y2, Example 4.11]. Let $(A, \mathfrak{m})$ be the same one as in Example (5.1). Then we have the following arguments too.

(i) $G(\mathfrak{m}^\alpha)$ is a Buchsbaum ring if and only if $\alpha \leqslant r$.
(ii) In particular, $G(\mathfrak{m}^r)$, $G(\mathfrak{m}^{r-1})$ are Buchsbaum rings, though $G_r(\mathfrak{m})$, $G_{r-1}(\mathfrak{m})$ are not Buchsbaum modules.

In fact, the case where $\alpha = 1$ is already done by (ii) of Example (5.1) above. So we may assume $\alpha \geqslant 2$. Under the same notations S, A, U, $\overline{A}$, etc. as in Example (5.1), we put $\mathfrak{a} := \mathfrak{m}^\alpha$ and $\bar{\mathfrak{a}} := \mathfrak{a}\overline{A}$. Let $\psi : G(\mathfrak{a}) \longrightarrow G(\bar{\mathfrak{a}})$ be the canonical epimorphism and put $U^* := \operatorname{Ker}\psi$. Since $G(\overline{\mathfrak{m}})$ is Cohen–Macaulay, $G(\bar{\mathfrak{a}})$ is also Cohen–Macaulay by [H-R] (cf. [Y3] too). Thus, by (2) of Lemma (3.5) we get $\mathrm{H}^0_M(G(\mathfrak{a})) = U^*$, hence

$$\left[\mathrm{H}^0_M\big(G(\mathfrak{a})\big)\right]_n = U \cap \mathfrak{a}^n / U \cap \mathfrak{a}^{n+1} \tag{$\sharp$5.6}$$

for all $n \in \mathbb{Z}$, here M denotes the unique homogeneous maximal ideal of $R(\mathfrak{a})$, i.e., $M := \mathfrak{m}R(\mathfrak{a}) + R(\mathfrak{a})_+$. We put $q := [r/\alpha]$, the Gauss symbol of the rational number r/α, namely q is the largest integer n such that $n \leqslant r/\alpha$. By our choices of α and q, we know that $\mathfrak{m}^2 \supseteq \mathfrak{a} \supseteq \mathfrak{a}^q \supseteq \mathfrak{m}^r$, hence we have a chain of A-submodules of U

$$U \supsetneq U \cap \mathfrak{a} = \cdots = U \cap \mathfrak{a}^q = \mathfrak{m}U \supsetneq U \cap \mathfrak{a}^{q+1} = (0). \tag{$\sharp$5.7}$$

Then we have

$$\left[\mathrm{H}^0_M\big(G(\mathfrak{a})\big)\right]_n = \begin{cases} U/\mathfrak{m}U & (n=0), \\ \mathfrak{m}U & (n=q), \\ (0) & (\text{else}) \end{cases} \tag{$\sharp$5.8}$$

by (♯5.6) and (♯5.7), where $q \geqslant 1$. At first, let $q \geqslant 2$. Then by (♯5.8), it is clear that $M \cdot \mathrm{H}^0_M(G(\mathfrak{a})) = (0)$. Next, let $q = 1$, i.e., $r/2 < \alpha \leqslant r$. Then, recalling $\mathfrak{a} \subseteq \mathfrak{m}^2$, we know $\mathfrak{a}U = (0)$ by calculations before. Hence we have $\mathfrak{a}t \cdot \mathrm{H}^0_M(G(\mathfrak{a})) = (0)$ by (♯5.8) again. So, we conclude that $M \cdot \mathrm{H}^0_M(G(\mathfrak{a})) = (0)$ in the case where $q \geqslant 1$. Now, let $q = 0$, i.e., $r < \alpha$. Then we have

$$[\mathrm{H}^0_M(G(\mathfrak{a}))]_n = \begin{cases} U & (n = 0), \\ (0) & (\text{else}), \end{cases}$$

by (♯5.6), (♯5.7) again. This means $\mathfrak{m} \cdot [\mathrm{H}^0_M(G(\mathfrak{a}))]_0 \neq (0)$, hence $M \cdot \mathrm{H}^0_M(G(\mathfrak{a})) \neq (0)$. So, we finally get the following:

$$M \cdot \mathrm{H}^0_M(G(\mathfrak{a})) = (0) \iff q \geqslant 1 \iff \alpha \leqslant r.$$

Therefore $G(\mathfrak{a})$ is a Buchsbaum ring if and only if $\alpha \leqslant r$. The remaining part of our assertions follows from (iv) of Example (5.1) at once.

Remark (5.3). We shall introduce two more topics concerning of our new graded module $G_\alpha(\mathfrak{a}, E)$. For conveniences of the readers, we recall the definition of them under more general situations.

Let A be a commutative ring, E an A-module and $\mathfrak{a}$ an ideal of A. We denote by $R(\mathfrak{a}) := \bigoplus_{n \geqslant 0} \mathfrak{a}^n$ the Rees algebra of $\mathfrak{a}$. For any integer $\alpha > 0$, we define a new graded module, writing $G_\alpha(\mathfrak{a}, E)$, as follows:

$$G_\alpha(\mathfrak{a}, E) := \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}^n E / \mathfrak{a}^{n+\alpha} E,$$

here we set $\mathfrak{a}^n = A$ for each $n \leqslant 0$. Then this $G_\alpha(\mathfrak{a}, E)$ becomes a $\mathbb{Z}$-graded module over $R(\mathfrak{a})$ by the graduation such that $[G_\alpha(\mathfrak{a}, E)]_n = \mathfrak{a}^n E / \mathfrak{a}^{n+\alpha} E$ where $n \in \mathbb{Z}$. In particular, $[G_\alpha(\mathfrak{a}, E)]_n = (0)$ for $n \leqslant -\alpha$ and $G_1(\mathfrak{a}, E) = G(\mathfrak{a}, E)$ clearly.

(1) Let $(*)^{(\alpha)}$ denote the *Veronesean* functor over $R(\mathfrak{a})$ of order $\alpha > 0$; namely, $R(\mathfrak{a})^{(\alpha)}$ is a graded ring whose graduation is defined by $[R(\mathfrak{a})^{(\alpha)}]_n = \mathfrak{a}^{\alpha n}$, hence $R(\mathfrak{a})^{(\alpha)} \cong R(\mathfrak{a}^\alpha)$ as graded A-algebras. Moreover, if $W := \bigoplus_{n \in \mathbb{Z}} W_n$ is a $\mathbb{Z}$-graded $R(\mathfrak{a})$-module, then $W^{(\alpha)}$ is a $\mathbb{Z}$-graded $R(\mathfrak{a}^\alpha)$-module, whose graduation is also defined by $[W^{(\alpha)}]_n = W_{\alpha n}$.

With this notation we have the following isomorphism

$$G_\alpha(\mathfrak{a}, E)^{(\alpha)} \cong G(\mathfrak{a}^\alpha, E)$$

as $\mathbb{Z}$-graded $R(\mathfrak{a}^\alpha)$-modules, because it is easy to see that

$$[G_\alpha(\mathfrak{a}, E)^{(\alpha)}]_n = \mathfrak{a}^{\alpha n} E / \mathfrak{a}^{\alpha(n+1)} E$$

for all $n \geqslant 0$; cf. [H-R-S, §2], and see [Y2] too.

This isomorphism tells us that the new graded module $G_\alpha(\mathfrak{a}, E)$ is very useful notion to studying properties of $G(\mathfrak{a}^\alpha, E)$, the associated graded module of E with respect to the power of the given ideal $\mathfrak{a}$.

(2) Assume that there is an element $a \in \mathfrak{a}$ such that a is a non-zero-divisor on E, i.e., $[0 :_E a] = (0)$, and that $aE \cap \mathfrak{a}^n E = a\mathfrak{a}^{n-1} E$ holds for all $n \geqslant 0$. Let $R(\mathfrak{a}, E) \longrightarrow R(\mathfrak{a}, E/a^\alpha E)$ be a canonical epimorphism of the Rees modules induced by $E \longrightarrow E/a^\alpha E$. It is obvious that $a^\alpha \cdot R(\mathfrak{a}, E)$ is contained in its kernel, hence it induces the canonical epimorphism $\rho : R(\mathfrak{a}, E)/a^\alpha \cdot R(\mathfrak{a}, E) \longrightarrow R(\mathfrak{a}, E/a^\alpha E)$ too. Then there exists an exact sequence of graded $R(\mathfrak{a})$-modules as follows:

$$0 \longrightarrow G_\alpha(\mathfrak{a}, E)(-\alpha) \longrightarrow R(\mathfrak{a}, E)/a^\alpha \cdot R(\mathfrak{a}, E) \xrightarrow{\rho} R(\mathfrak{a}, E/a^\alpha E) \longrightarrow 0.$$

In fact, by our assumption on the element a it holds that $a^\alpha E \cap \mathfrak{a}^n E = a^\alpha \mathfrak{a}^{n-\alpha} E \cong \mathfrak{a}^{n-\alpha} E$ for all $n \in \mathbb{Z}$, via the multiplication by a^α. Therefore we get that

$$[\operatorname{Ker} \rho]_n = \mathfrak{a}^n E \cap a^\alpha E / a^\alpha \mathfrak{a}^n E \cong \mathfrak{a}^{n-\alpha} E / \mathfrak{a}^n E = \left[G_\alpha(\mathfrak{a}, E)(-\alpha)\right]_n.$$

This exact sequence inspires us that our new graded module $G_\alpha(\mathfrak{a}, E)$ might play important roles to calculating the Rees module $R(\mathfrak{a}, E)$ under suitable situations.

Acknowledgments

The author was supported by Grant-in-Aid for Scientific Research (C) in Japan, Project No. 17540051 (2005/2008).

References

[G1] S. Goto, Buchsbaum rings of maximal embedding dimension, J. Algebra 76 (1982) 383–399.
[G2] S. Goto, On the associated graded rings of parameter ideals in Buchsbaum rings, J. Algebra 85 (1983) 490–534.
[G3] S. Goto, Noetherian local rings with Buchsbaum associated graded rings, J. Algebra 86 (1984) 336–384.
[G-Ni] S. Goto, K. Nishida, Hilbert coefficients and Buchsbaumness of associated graded rings, J. Pure Appl. Algebra 181 (2003) 61–74.
[G-W] S. Goto, K. Watanabe, On graded rings, I, J. Math. Soc. Japan 30 (1978) 179–213.
[G-Y] S. Goto, K. Yamagishi, The theory of unconditioned strong d-sequences and modules of finite local cohomology, preprint.
[H-R] M. Hochster, L.J. Ratliff Jr., Five theorems on Macaulay rings, Pacific J. Math. 44 (1973) 147–172.
[H-R-S] M. Herrmann, J. Ribbe, P. Schenzel, On the Gorenstein property of form rings, Math. Z. 213 (1993) 301–309.
[H] C. Huneke, The theory of d-sequences and powers of ideals, Adv. Math. 46 (1982) 249–279.
[Na] Y. Nakamura, On the Buchsbaum property of associated graded rings, J. Algebra 209 (1998) 345–366.
[S] P. Schenzel, Standard systems of parameters and their blow-up rings, J. Reine Angew. Math. 344 (1983) 201–220.
[S-T-C] P. Schenzel, N.V. Trung, N.T. Cuong, Verallgemeinerte Cohen–Macaulay-Moduln, Math. Nachr. 85 (1978) 57–73.
[Su] N. Suzuki, On a basic theorem for quasi-Buchsbaum modules, Bull. Dept. Gen. Ed. Shizuoka College of Pharmacy 11 (1982) 33–40.
[S-V] J. Stückrad, W. Vogel, Buchsbaum Rings and Applications, Springer-Verlag, Berlin, New York, Tokyo, 1986.
[T] N.V. Trung, Toward a theory of generalized Cohen–Macaulay modules, Nagoya Math. J. 102 (1986) 1–49.
[Y1] K. Yamagishi, Idealizations of Buchsbaum modules over a Buchsbaum ring, Math. Proc. Cambridge Philos. Soc. 104 (1988) 451–478.
[Y2] K. Yamagishi, The associated graded modules of Buchsbaum modules with respect to $\mathfrak{m}$-primary ideals in the equi-$\mathbb{I}$-invariant case, J. Algebra 225 (2000) 1–27.
[Y3] K. Yamagishi, Asymptotic property of the $\mathbb{I}$-invariant of the associated graded modules, Comm. Algebra 31 (2003) 1031–1043.